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On some Gaussian ensembles of Hermitian matrices

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Abstract. A Gaussian ensemble of Hermitian matrices with an arbitrary ratio of their symmetric and anti-symmetric parts is studied. Similarly a Gaussian ensemble of Hermitian quaternion matrices with an arbitrary ratio of their self-dual and anti-self-dual parts is studied. Analytical expressions for the correlation and cluster functions, as well as their limits when the order of the matrices is large, are derived for both ensembles.

1. Introduction

The problem of the distribution of eigenvalues of random matrices arose in connection with the distribution of slow neutron resonance levels in nuclear spectra (Porter 1965, Wigner 1967, Mehta 1967, Brody *et al* 1981). The types of matrix ensembles to be considered depend on whether or not nuclear forces are invariant under time reversal. Three particular ensembles have been extensively studied.

(i) The ensemble of real symmetric matrices, known as the Gaussian orthogonal ensemble (GOE).

(ii) The ensemble of Hermitian matrices with equally probable real and imaginary parts (for quaternion matrices, equally probable quaternion real and quaternion imaginary parts) known as the Gaussian unitary ensemble (GUE).

(iii) The ensemble of self-dual quaternion matrices known as the Gaussian symplectic ensemble (GSE).

These three ensembles are basic models for energy level fluctuations of complex systems. For time-reversal-invariant systems, GOE or GSE is appropriate, depending on the properties of the Hamiltonian (Dyson 1962a, Mehta 1967 ch 2). On the other hand in the absence of time-reversal symmetry, the GUE is valid. For nuclear spectra GOE appears to be a good model (see Haq *et al* (1982) for a recent comparison of theory and experiment).

Ensembles with an arbitrary ratio of time-reversal invariant and non-invariant parts have been of renewed interest in the past few years (Pandey 1981, French *et al* 1983, Pandey and Mehta 1983). The motivation underlying these studies is the suggestion, due to Wigner (1967), that the analysis of data in comparison with such ensembles may give an upper bound on the time-reversal breaking part of the nuclear forces. These studies reveal that the transition in the fluctuation properties, for the ensembles going from GOE to GUE, adequate for the above purpose, is very rapid and, in fact, discontinuous for infinite-order matrices. Available results (Pandey 1981, French *et al* 1983) indicate that, for infinite-order matrices, the GUE results are valid even for ensembles in which the imaginary part is larger in magnitude than the real

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part (see Mehta and Rosenzweig (1968), for the extreme case of purely imaginary matrices). Similar results were expected for ensembles going from GSE to GUE and beyond.

In an earlier paper (Pandey and Mehta 1983) we considered in detail ensembles intermediate between GOE and GUE. In particular, we derived all correlation and cluster properties of the eigenvalues for finite- as well as infinite-order matrices. A similar analysis works for ensembles intermediate between GSE and GUE. In both cases, with minor changes, the results apply equally well for ensembles beyond GUE, including the extreme cases of anti-symmetric and anti-self-dual matrices. Though not relevant for immediate physical applications, these results are obtained with little extra effort and it is worthwhile reporting them for their mathematical elegance.

In § 2 we describe the ensembles and give a summary of the results. Later sections deal with the proofs; in § 3 we derive the joint probability density for the eigenvalues and in § 4 the correlation and cluster functions for finite- as well as infinite-order matrices. Several appendices follow; appendix 1 contains recurrence relations and integrals which are frequently used in this article; in appendices 2 and 3 the normalisation integrals for the joint probability density of the eigenvalues is derived and the last two appendices contain the verification of some equations of § 4.

2. Summary of results

2.1. Matrix ensembles from GOE to GUE and beyond

Consider an ensemble of $N \times N$ Hermitian matrices $[H_{jk}] = [R_{jk} + \mathrm{i}S_{jk}]$, with R real symmetric and S real anti-symmetric, i.e.

$$R_{jk} = R^*_{jk} = R_{kj}, \qquad S_{jk} = S^*_{jk} = -S_{kj}. \tag{2.1}$$

The joint probability density for the matrix elements is taken to be

$$P(H) = c \exp\left[-\sum_{j,k}\left(\frac{R^2_{jk}}{4v^2} + \frac{S^2_{jk}}{4v^2\alpha^2}\right)\right], \tag{2.2}$$

$$\mathrm{d}H = \prod_{j\leqslant k} \mathrm{d}R_{jk} \prod_{j<k} \mathrm{d}S_{jk}, \tag{2.3}$$

where the normalisation constant c is

$$c = 2^{-N/2}\alpha^{-N(N-1)/2}(2\pi v^2)^{-N^2/2}. \tag{2.4}$$

On average

$$\frac{\langle\|\mathrm{Im}\, H\|^2\rangle}{\langle\|\mathrm{Re}\, H\|^2\rangle} = \frac{\langle\|S\|^2\rangle}{\langle\|R\|^2\rangle} = \frac{N-1}{N+1}\alpha^2 \approx \alpha^2 \qquad \text{for large } N. \tag{2.5}$$

We shall choose the scale v^2 such that

$$2v^2(1+\alpha^2) = 1. \tag{2.6}$$

As special cases we have; (i) $\alpha^2 = 0$, so that $S = 0$ with probability one and the matrices H form the GOE; (ii) $\alpha^2 = 1$, on the average R and S have the same magnitude for large N and the ensemble is GUE; (iii) $\alpha^2 \to \infty$, S dominates R, and the ensemble of H may be referred to as the anti-symmetric Gaussian orthogonal ensemble (AGOE).

The joint probability density for the eigenvalues $x_1, \ldots, x_N$ of H is

$$p(x_1, \ldots, x_N) = C_N \exp\left[-\tfrac{1}{2}(1+\alpha^2) \sum x_j^2\right] \Delta(x_1, \ldots, x_N) \, \mathrm{Pf}[F_{ij}], \tag{2.7}$$

where

$$\Delta(x) \equiv \Delta(x_1, \ldots, x_N) = \prod_{1 \leq i < j \leq N} (x_i - x_j), \tag{2.8}$$

and Pf $[F_{ij}]$ is the Pfaffian of a $2m \times 2m$ matrix defined below. If N is even, $2m = N$, we define

$$F_{ij} = f(x_i - x_j), \qquad i, j = 1, 2, \ldots, N. \tag{2.9}$$

If N is odd, $2m = N+1$, we use the preceding definition and in addition

$$F_{i,N+1} = -F_{N+1,i} = 1, \qquad i = 1, 2, \ldots, N, \qquad F_{N+1,N+1} = 0 \tag{2.9a}$$

with

$$f(x) = \mathrm{erf}\left[\left(\frac{1-\alpha^4}{4\alpha^2}\right)^{1/2} x\right] \equiv \left(\frac{1-\alpha^4}{\pi\alpha^2}\right)^{1/2} \int_0^x \exp\left(-\frac{1-\alpha^4}{4\alpha^2} y^2\right) \mathrm{d}y. \tag{2.10}$$

The $[F_{ij}]$ is an anti-symmetric matrix of even order and its Pfaffian† is, apart from a sign, the square root of its determinant. The normalisation constant C_N is

$$C_N^{-1} = 2^{3N/2}(1-\alpha^2)^{N(N-1)/4}(1+\alpha^2)^{-N(N+1)/4} \prod_{j=1}^{N} \Gamma(1+\tfrac{1}{2}j). \tag{2.11}$$

Note that when $\alpha^2 > 1$, $f(x)$ and C_N are both pure imaginary but $p(x_1, \ldots, x_N)$ remains real positive. Moreover, for $\alpha^2 = 0$, 1 or ∞, $p(x_1, \ldots, x_N)$ and all the quantities derived below have well defined limits.

All the eigenvalue correlations can be expressed in terms of functions of two variables. The n-level correlation function (Dyson 1962a, Mehta 1967 ch 5.1) is

$$R_n(x_{i_1}, \ldots, x_n) = \{\det[\Phi(x_i, x_j)]_{i,j=1,\ldots,n}\}^{1/2} \tag{2.12}$$

and the n-level cluster function (cumulant of the preceding) is

$$T_n(x_1, \ldots, x_n) = \tfrac{1}{2} \mathrm{Tr}\, \Sigma\, \Phi(x_1, x_2)\Phi(x_2, x_3) \ldots \Phi(x_n, x_1), \tag{2.13}$$

where

$$\Phi(x, y) = \begin{bmatrix} \xi_N(x, y) + S_N(x, y) & D_N(x, y) \\ J_N(x, y) & \xi_N(y, x) + S_N(y, x) \end{bmatrix} \tag{2.14}$$

and the sum in (2.13) is taken over all $(n-1)!$ distinct cyclic permutations of the

† The Pfaffian (see e.g. Mehta 1977) of any $2m \times 2m$ anti-symmetric matrix $A = [a_{ij}]$ is the alternating multi-linear form in the elements a_{ij} with $i < j$,

$$\mathrm{Pf}\, A \equiv \mathrm{Pf}[a_{ij}] = \sum \pm a_{i_1 i_2} a_{i_3 i_4} \ldots a_{i_{2m-1} i_{2m}},$$

where the sum is taken over all $(2m)!/(2^m m!)$ permutations $(i_1, i_2, \ldots, i_{2m})$ of $(1, 2, \ldots, 2m)$ with the restrictions $i_1 < i_2$, $i_3 < i_4, \ldots, i_{2m-1} < i_{2m}$, $i_1 < i_3 < \ldots < i_{2m-1}$, and the sign is + or − according to whether the permutation is even or odd.

indices $(1, 2, \ldots, n)$. Note the interchange of x and y in the lower right-hand corner of (2.14). The two-point functions ξ_N, S_N, D_N and J_N are given in terms of other two-point functions $I_N(x, y)$, $g(x, y)$, $\mu_N(x, y)$ and the one-point functions $\varphi_j(x)$, $\psi_j(x)$, $A_j(x)$, and $\varepsilon(x)$ defined below.

$$\xi_N(x, y) = \begin{cases} \varphi_{2m}(x) \exp(-\tfrac{1}{2}\alpha^2 y^2) \Big/ \int_{-\infty}^{\infty} \varphi_{2m}(t) \exp(-\tfrac{1}{2}\alpha^2 t^2)\, dt, & N = 2m+1 \text{ odd} \\ 0, & N = 2m \text{ even}, \end{cases} \tag{2.15}$$

$$S_N(x, y) = \sum_{j=0}^{N-1} \varphi_j(x)\varphi_j(y) + (1-\alpha^2)\left(\frac{1+\alpha^2}{1-\alpha^2}\right)^N (\tfrac{1}{2}N)^{1/2} \varphi_{N-1}(x) A_N(y) \tag{2.16}$$

$$= \begin{cases} \sum_{j=0}^{m-1} [\varphi_{2j}(x)\varphi_{2j}(y) - \psi_{2j}(x) A_{2j}(y)], & N = 2m \text{ even}, \\ \sum_{j=0}^{m-1} [\varphi_{2j+1}(x)\varphi_{2j+1}(y) - \psi_{2j+1}(x) A_{2j+1}(y)], & N = 2m+1 \text{ odd}, \end{cases} \tag{2.17}$$

$$D_N(x, y) = \sum_{j=0}^{N-1} \varphi_j(x)\psi_j(y) + (1-\alpha^2)\left(\frac{1+\alpha^2}{1-\alpha^2}\right)^N (\tfrac{1}{2}N)^{1/2} \varphi_{N-1}(x)\varphi_N(y) \tag{2.18}$$

$$= \begin{cases} \sum_{j=0}^{m-1} [\varphi_{2j}(x)\psi_{2j}(y) - \psi_{2j}(x)\varphi_{2j}(y)], & N = 2m \text{ even}, \\ \sum_{j=0}^{m-1} [\varphi_{2j+1}(x)\psi_{2j+1}(y) - \psi_{2j+1}(x)\varphi_{2j+1}(y)], & N - 2m+1 \text{ odd}, \end{cases} \tag{2.19}$$

$$J_N(x, y) = I_N(x, y) + g(x, y) + \mu_N(x, y) - \mu_N(y, x), \tag{2.20}$$

where

$$I_N(x, y) = \sum_{j=0}^{N-1} \varphi_j(x) A_j(y) + (1-\alpha^2)[(1+\alpha^2)/(1-\alpha^2)]^N (\tfrac{1}{2}N)^{1/2} A_N(x) A_{N-1}(y) \tag{2.21}$$

$$= \begin{cases} \sum_{j=0}^{m-1} [\varphi_{2j}(x) A_{2j}(y) - A_{2j}(x)\varphi_{2j}(y)], & N = 2m \text{ even}, \\ \sum_{j=0}^{m-1} [\varphi_{2j+1}(x) A_{2j+1}(y) - A_{2j+1}(x)\varphi_{2j+1}(y)] & N = 2m+1 \text{ odd}, \end{cases} \tag{2.22}$$

$$g(x, y) = \tfrac{1}{2}[(1+\alpha^2)/(1-\alpha^2)]^{1/2} \exp[-\tfrac{1}{2}\alpha^2(x^2+y^2)] f(x-y), \tag{2.23}$$

$$= \sum_{j=0}^{\infty} A_j(x)\varphi_j(y) = \tfrac{1}{2}\sum_{j=0}^{\infty} [A_j(x)\varphi_j(y) - \varphi_j(x) A_j(y)]$$

$$= \sum_{j=0}^{\infty} [A_{2j}(x)\varphi_{2j}(y) - \varphi_{2j}(x) A_{2j}(y)]$$

$$= \sum_{j=0}^{\infty} [A_{2j+1}(x)\varphi_{2j+1}(y) - \varphi_{2j+1}(x) A_{2j+1}(y)], \tag{2.24}$$

$$\mu_N(x,y)=\begin{cases}\exp(-\tfrac{1}{2}\alpha^2x^2)A_{2m}(y)\Big/\int_{-\infty}^{\infty}\varphi_{2m}(t)\exp(-\tfrac{1}{2}\alpha^2t^2)\,\mathrm{d}t, & N=2m+1 \text{ odd},\\ 0, & N=2m \text{ even},\end{cases} \tag{2.25}$$

$$\varphi_j(x)=(2^j j!\sqrt{\pi})^{-1/2}\exp(\tfrac{1}{2}x^2)(-\mathrm{d}/\mathrm{d}x)^j\exp(-x^2), \tag{2.26}$$

$$\psi_j(x)=[(1+\alpha^2)/(1-\alpha^2)]^j\exp(\tfrac{1}{2}\alpha^2x^2)\,(\mathrm{d}/\mathrm{d}x)[\exp(-\tfrac{1}{2}\alpha^2x^2)\varphi_j(x)], \tag{2.27}$$

$$A_j(x)=[(1-\alpha^2)/(1+\alpha^2)]^j\exp(-\tfrac{1}{2}\alpha^2x^2)\int_{-\infty}^{\infty}\exp(\tfrac{1}{2}\alpha^2t^2)\varepsilon(x-t)\varphi_j(t)\,\mathrm{d}t, \tag{2.28}$$

$$\varepsilon(x)=\tfrac{1}{2}\operatorname{sgn}x. \tag{2.29}$$

For large N, the level density $R_1(x)$ becomes a semi-circle for all α

$$R_1(x)\approx\pi^{-1}(2N-x^2)^{1/2}. \tag{2.30}$$

The n-level correlation and cluster functions for $n>1$ are discontinuous in α^2 at $\alpha^2=0$. We have the GOE results for $\alpha^2=0$ and the GUE results for $\alpha^2>0$. On the other hand if

$$\lambda\equiv\lambda(x)=\frac{\text{RMS value of }(\operatorname{Im}, H_{ij})}{\text{Local average spacing at }x}\approx\frac{\alpha}{\sqrt{2}}R_1(x) \tag{2.31}$$

remains finite when $\alpha^2\to0$ and $N\to\infty$, the n-level correlation and cluster functions have well defined limits for all λ. With $x_i-x_j\to0$ and $(x_i-x_j)R(x_i)\to r_{ij}=r_i-r_j$, the n-level correlation function becomes

$$\begin{aligned}R_n(r_1,\ldots,r_n;\lambda)&\equiv\lim_{N\to\infty}\{R_1(x_1),\ldots,R_1(x_n)\}^{-1}R_n(x_1,\ldots,x_n)\\&=\{\det[\sigma(r_{ij};\lambda)]_{i,j=1,\ldots,n}\}^{1/2}\end{aligned} \tag{2.32}$$

and the corresponding n-level cluster function is

$$\begin{aligned}Y_n(r_1,\ldots,r_n;\lambda)&\equiv\lim_{N\to\infty}\{R_1(x_1)\ldots R_1(x_n)\}^{-1}T_n(x_1,\ldots,x_n)\\&=\tfrac{1}{2}\operatorname{Tr}\sum\sigma(r_{12};\lambda)\sigma(r_{23};\lambda)\ldots\sigma(r_{n1};\lambda)\end{aligned} \tag{2.33}$$

where the sum is taken, as in (2.13), over all $(n-1)!$ distinct cyclic permutations of the indices $(1,2,\ldots,n)$ and

$$\sigma(r;\lambda)=\begin{bmatrix}(\sin\pi r)/\pi r & D(r;\lambda)\\ J(r;\lambda) & (\sin\pi r)/\pi r\end{bmatrix}, \tag{2.34}$$

with

$$D(r;\lambda)=-\frac{1}{\pi}\int_0^{\pi}k\sin kr\exp(2\lambda^2k^2)\,\mathrm{d}k, \tag{2.35}$$

$$J(r;\lambda)=-\frac{1}{\pi}\int_{\pi}^{\infty}\frac{\sin kr}{k}\exp(-2\lambda^2k^2)\,\mathrm{d}k. \tag{2.36}$$

Note that as $\lambda\to\infty$, $D\to\infty$, $J\to0$, while the product $JD\to0$, so that in the expressions for R_n and Y_n one may replace D and J by zeros.

2.2. Matrix ensembles from GSE to GUE and beyond

Introducing the quaternion units

$$1=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \qquad e_1=\begin{bmatrix}0 & \mathrm{i}\\ \mathrm{i} & 0\end{bmatrix}, \qquad e_2=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}, \qquad e_3=\begin{bmatrix}\mathrm{i} & 0\\0 & -\mathrm{i}\end{bmatrix}, \tag{2.37}$$

we shall think of a quaternion

$$q=q_0+q_1e_1+q_2e_2+q_3e_3=\begin{bmatrix}q_0+\mathrm{i}q_3 & \mathrm{i}q_1-q_2\\ \mathrm{i}q_1+q_2 & q_0-\mathrm{i}q_3\end{bmatrix} \tag{2.38}$$

as a 2×2 matrix. Conversely any 2×2 matrix will be taken as a quaternion with q_0, q_1, q_2, q_3 as ordinary complex numbers†. An $N\times N$ matrix with quaternion elements is thus equivalent to an ordinary $2N\times 2N$ matrix, and conversely any $2N\times 2N$ matrix can be cut into N^2 blocks of 2×2 and each 2×2 block will be regarded as a quaternion. For definitions and properties pertaining to quaternion matrices see Dyson (1970, 1972), Mehta (1977 ch 8). We recall in particular that a quaternion has two types of conjugates:

$$\bar{q}=q_0-q_1e_1-q_2e_2-q_3e_3 \tag{2.39}$$

and

$$q^*=q_0^*+q_1^*e_1+q_2^*e_2+q_3^*e_3. \tag{2.40}$$

The two conjugates together give the Hermitian conjugate

$$q^\dagger=q_0^*-q_1^*e_1-q_2^*e_2-q_3^*e_3. \tag{2.41}$$

The quaternion q is called real if $q=q^*$ and scalar if $q=\bar{q}$. Note that the 2×2 matrix representative of a real quaternion does not necessarily have real elements. A matrix M with quaternion elements M_{jk} is 'quaternion real' if $M_{jk}=M_{jk}^*$; 'quaternion pure imaginary' if $M_{jk}=-M_{jk}^*$; self-dual if $\bar{M}_{jk}=M_{kj}$, anti-self-dual if $\bar{M}_{jk}=-M_{kj}$ and Hermitian if $M_{jk}^\dagger=M_{kj}$. The matrix M is unitary if $MM^\dagger=M^\dagger M=1$ and it is symplectic if $M\bar{M}=\bar{M}M=1$. A Hermitian self-dual matrix M can be diagonalised by a unitary symplectic matrix U

$$M=UDU^\dagger, \qquad UU^\dagger=U\bar{U}=1, \tag{2.42}$$

where the diagonal elements of D are real and scalar.

Now, as in § 2.1, consider an ensemble of $N\times N$ Hermitian quaternion matrices

$$[\boldsymbol{H}_{jk}]=[\boldsymbol{R}_{jk}+\mathrm{i}\boldsymbol{S}_{jk}], \tag{2.43}$$

where $\boldsymbol{R}$ and $\boldsymbol{S}$ are real self-dual and real anti-self-dual quaternion matrices respectively, i.e. with

$$\boldsymbol{R}=\boldsymbol{R}^0+\sum_{\mu=1}^{3}\boldsymbol{R}^\mu e_\mu, \qquad \boldsymbol{S}=\boldsymbol{S}^0+\sum_{\mu=1}^{3}\boldsymbol{S}^\mu e_\mu; \tag{2.44}$$

$\boldsymbol{R}^0$ and $\boldsymbol{S}^\mu$ are real symmetric while $\boldsymbol{R}^\mu$ and $\boldsymbol{S}^0$ are real anti-symmetric. The probability

† The coefficients q_0, q_1, q_2, and q_3 are usually taken to be real numbers so that the quaternions may form a field. We are relaxing this constraint since we do not need the property that every non-zero quaternion must have an inverse.

density for the matrix elements will be taken as

$$\boldsymbol{P}(\boldsymbol{H}) = \boldsymbol{c}\,\exp\left[-\sum_{j,k}\sum_{\mu=0}^{3}\left(\frac{(\boldsymbol{R}_{jk}^{\mu})^2}{4v^2} + \frac{(\boldsymbol{S}_{jk}^{\mu})^2}{4v^2\alpha^2}\right)\right] \tag{2.45}$$

$$\mathrm{d}\boldsymbol{H} = \prod_{j\leqslant k}\left(\mathrm{d}\boldsymbol{R}_{jk}^{0}\prod_{\mu=1}^{3}\mathrm{d}\boldsymbol{S}_{jk}^{\mu}\right)\prod_{j<k}\left(\mathrm{d}\boldsymbol{S}_{jk}^{0}\prod_{\mu=1}^{3}\mathrm{d}\boldsymbol{R}_{jk}^{\mu}\right), \tag{2.46}$$

with the normalisation constant

$$\boldsymbol{c} = 2^{-2N}\alpha^{-N(2N+1)}(2\pi v^2)^{-2N^2}. \tag{2.47}$$

On average the ratio of the anti-self-dual and self-dual parts of $\boldsymbol{H}$ is

$$\langle\|\boldsymbol{S}\|^2\rangle/\langle\|\boldsymbol{R}\|^2\rangle = [(2N+1)/(2N-1)]\alpha^2 \approx \alpha^2 \qquad \text{for large } N. \tag{2.48}$$

We shall again choose the scale $2v^2(1+\alpha^2) = 1$ as in (2.6).

As previously, we have the particular cases: (i) $\alpha^2 = 0$, so that $\boldsymbol{S} = 0$ with probability one and the matrices $\boldsymbol{H}$ form the GSE; (ii) $\alpha^2 = 1$, on the average $\boldsymbol{R}$ and $\boldsymbol{S}$ have the same magnitude and the ensemble is GUE (of $2N\times 2N$ matrices); (iii) $\alpha^2 \to \infty$, $\boldsymbol{S}$ dominates $\boldsymbol{R}$ and the ensemble of $\boldsymbol{H}$ may be referred to as the anti-self-dual Gaussian symplectic ensemble (AGSE).

The joint probability density for the eigenvalues $x_1, \ldots, x_{2N}$ of $\boldsymbol{H}$ is

$$\boldsymbol{p}(x_1, \ldots, x_{2N}) = \boldsymbol{C}_N \exp\left(-\frac{1+\alpha^2}{2}\sum x_j^2\right)\Delta(x_1, \ldots, x_{2N})\,\mathrm{Pf}[\boldsymbol{F}(x_i - x_j)]_{i,j=1,\ldots,2N}, \tag{2.49}$$

where Δ is the product of differences of the $2N$ variables $x_1, \ldots, x_{2N}$, equation (2.8),

$$\boldsymbol{F}(x) = x\,\exp[-(1-\alpha^4)x^2/4\alpha^2] \tag{2.50}$$

and

$$\boldsymbol{C}_N^{-1} = 2^{-N(N-1)}\pi^N\alpha^{3N}(1-\alpha^2)^{N(N-1)}(1+\alpha^2)^{-N(N+1)}(2N)!\,(N!)^{-1}\prod_{1}^{N}(2j)! \tag{2.51}$$

The limits in the three cases $\alpha^2 = 0$, 1 and ∞ are again well defined.

The n-level correlation functions and their cumulants are again given by (2.12) and (2.13) with Φ replaced by

$$\boldsymbol{\Phi}(x, y) = \begin{bmatrix}\boldsymbol{S}_N(x, y) & \boldsymbol{K}_N(x, y)\\ \boldsymbol{I}_N(x, y) & \boldsymbol{S}_N(y, x)\end{bmatrix} \tag{2.52}$$

with the new functions

$$\boldsymbol{S}_N(x, y) = \sum_{j=0}^{N-1}[\varphi_{2j+1}(x)\varphi_{2j+1}(y) - \psi_{2j+1}(x)\boldsymbol{A}_{2j+1}(y)] \tag{2.53}$$

$$= \sum_{j=0}^{2N}\varphi_j(x)\varphi_j(y) + (1+\alpha^2)\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{2N+1}(N+\tfrac{1}{2})^{1/2}\varphi_{2N}(x)\boldsymbol{A}_{2N+1}(y), \tag{2.54}$$

$$\boldsymbol{I}_N(x, y) = \sum_{j=0}^{N-1}[\varphi_{2j+1}(x)\boldsymbol{A}_{2j+1}(y) - \boldsymbol{A}_{2j+1}(x)\varphi_{2j+1}(y)] \tag{2.55}$$

$$= \sum_{j=0}^{2N}\varphi_j(x)\boldsymbol{A}_j(y) + (1+\alpha^2)\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{2N+1}(N+\tfrac{1}{2})^{1/2}\boldsymbol{A}_{2N+1}(x)\boldsymbol{A}_{2N}(y), \tag{2.56}$$

$$\boldsymbol{K}_N(x, y) = \boldsymbol{D}_N(x, y) + \boldsymbol{g}(x, y), \tag{2.57}$$

$$\boldsymbol{D}_N(x, y) = \sum_{j=0}^{N-1} [\varphi_{2j+1}(x)\boldsymbol{\psi}_{2j+1}(y) - \boldsymbol{\psi}_{2j+1}(x)\varphi_{2j+1}(y)] \tag{2.58}$$

$$= \sum_{j=0}^{2N} \varphi_j(x)\boldsymbol{\psi}_j(y) + (1+\alpha^2)\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{2N+1} (N+\tfrac{1}{2})^{1/2}\varphi_{2N}(x)\varphi_{2N+1}(y), \tag{2.59}$$

$$\boldsymbol{g}(x, y) = -\frac{(1+\alpha^2)(1-\alpha^4)}{4\alpha^3\sqrt{\pi}} \exp[-\tfrac{1}{2}\alpha^2(x^2+y^2)]\boldsymbol{F}(x-y) \tag{2.60}$$

$$= \sum_{j=0}^{\infty} \boldsymbol{\psi}_j(x)\varphi_j(y) = \tfrac{1}{2}\sum_{j=0}^{\infty} [\boldsymbol{\psi}_j(x)\varphi_j(y) - \varphi_j(x)\boldsymbol{\psi}_j(y)]$$

$$= \sum_{j=0}^{\infty} [\boldsymbol{\psi}_{2j}(x)\varphi_{2j}(y) - \varphi_{2j}(x)\boldsymbol{\psi}_{2j}(y)]$$

$$= \sum_{j=0}^{\infty} [\boldsymbol{\psi}_{2j}(x)\varphi_{2j}(y) - \varphi_{2j}(x)\boldsymbol{\psi}_{2j+1}(y)] \tag{2.61}$$

where $\varphi_j(x)$ and $\varepsilon(x)$ are given by (2.26), (2.29),

$$\boldsymbol{\psi}_j(x) = \left(\frac{1-\alpha^2}{1+\alpha^2}\right)^j \exp(-\tfrac{1}{2}\alpha^2x^2)\frac{\mathrm{d}}{\mathrm{d}x}(\exp(\tfrac{1}{2}\alpha^2x^2)\varphi_j(x)), \tag{2.62}$$

$$\boldsymbol{A}_j(x) = \left(\frac{1+\alpha^2}{1-\alpha^2}\right)^j \exp(\tfrac{1}{2}\alpha^2x^2)\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}\alpha^2y^2)\varepsilon(x-y)\varphi_j(y)\,\mathrm{d}y. \tag{2.63}$$

For large N, we again have a semi-circular level density

$$\boldsymbol{R}_1(x) \approx \pi^{-1}(4N - x^2)^{1/2} \tag{2.64}$$

whereas the n-level functions for $n > 1$ are discontinuous in α^2 at $\alpha^2 = 0$. On the other hand when $\alpha \to 0$ and

$$\lambda \equiv \lambda(x) \approx (\alpha/\sqrt{2})\boldsymbol{R}_1(x) \tag{2.65}$$

is finite, the functions have well defined limits. Equations (2.32) and (2.33) are unchanged but with σ replaced by

$$\boldsymbol{\sigma}(r;\lambda) = \begin{bmatrix} (\sin\pi r)/\pi r & \boldsymbol{K}(r;\lambda) \\ \boldsymbol{I}(r;\lambda) & (\sin\pi r)/\pi r \end{bmatrix} \tag{2.66}$$

where

$$\boldsymbol{I}(r;\lambda) = -\pi^{-1}\int_0^{\pi} \frac{\sin kr}{k} \exp(2\lambda^2k^2)\,\mathrm{d}k, \tag{2.67}$$

$$\boldsymbol{K}(r;\lambda) = -\pi^{-1}\int_{\pi}^{\infty} k \sin kr \exp(-2\lambda^2k^2)\,\mathrm{d}k. \tag{2.68}$$

Note that the limit of $\boldsymbol{K}(r;\lambda)$ at $\lambda = 0$ contains a term in $\delta(r)/r$. This is due to the fact that in the GSE the eigenvalues are doubly degnerate. For $\lambda \to \infty$, the product IK being zero, I and K can be replaced by zeros (see the remark following (2.36)).

3. Joint probability density for the eigenvalues

In this section we derive the joint probability density for the eigenvalues, equations (2.7) and (2.49), from that of the matrix elements, equations (2.2) and (2.45). The matrix element probability densities depend on the eigenvalues and the angular variables characterising the eigenvectors, and one has to integrate over these angular variables. For $\alpha^2 = 0, 1, \infty$ in both ensembles the matrix element probability densities depend only on the eigenvalues. Also the Jacobian separates into a product of two functions, one involving only the eigenvalues and the other involving only the eigenvectors (Dyson 1962b, Mehta 1967 ch 3); therefore the integral over the angular variables, giving only a constant, need not be calculated. For arbitrary α^2 this simplification is not there, but we know (Itzykson and Zuber 1980, Mehta 1981) an integral over the group of unitary matrices U,

$$\int \exp[c\ \mathrm{Tr}(A - U^{+}BU)^2]\,\mathrm{d}U = \mathrm{constant}[\Delta(a)\Delta(b)]^{-1} \det\{\exp[c(a_i - b_j)^2]\} \tag{3.1}$$

valid for arbitrary Hermitian matrices A and B having eigenvalues $\{a_i\}$ and $\{b_i\}$ respectively, where Δ is the product of differences, equation (2.8). We also have two other results at our disposal.

(i) The convolution of two independent Gaussian distributions with variances σ_1^2 and $\sigma_2{}^2$ is again a Gaussian with the variance $\sigma_1^2 + \sigma_2^2$:

$$\int_{-\infty}^{\infty} (2\pi\sigma_1^2)^{-1/2} \exp\left(-\frac{x^2}{2\sigma_1^2}\right)(2\pi\sigma_2^2)^{-1/2} \exp\left(-\frac{(y-x)^2}{2\sigma_2^2}\right) \mathrm{d}x$$
$$= [2\pi(\sigma_1^2 + \sigma_2^2)]^{-1/2} \exp[-y^2/2(\sigma_1^2 + \sigma_2^2)], \tag{3.2}$$

(ii) For any sets of functions $\theta_i(x)$, $\tau_i(x)$ and $\chi_i(x)$ and for a suitable measure $\mathrm{d}\mu(x)$, integrals of the form

$$I_1 = \int \ldots \int \prod_1^N \mathrm{d}\mu(x_i) \det[\theta_i(x_j)]\ \mathrm{sgn}\ \Delta(x), \tag{3.3}$$

$$I_2 = \int \ldots \int \prod_1^m \mathrm{d}\mu(x_i) \det\,[\theta_i(x_j),\ \tau_i(x_j)]_{\substack{i=1,\ldots,2m\\ j=1,\ldots,m}} \tag{3.4}$$

$$I_3 = \int \ldots \int \prod_1^m \mathrm{d}\mu(x_i) \det[\theta_i(x_j),\ \tau_i(x_j),\ \chi_i(x_m)]_{\substack{i=1,\ldots,2m-1\\ j=1,\ldots,m-1}} \tag{3.5}$$

can be evaluated by the method of integration over alternate variables and the theory of Pfaffians (Mehta 1967 ch 5.2 and appendix A.7). The result is

$$I_1 = N!\,\mathrm{Pf}[a_{ij}]_{i,j=1,\ldots,2m}, \tag{3.6}$$

$$I_2 = m!\ \mathrm{Pf}[b_{ij}]_{i,j=1,\ldots,2m}, \tag{3.7}$$

$$I_3 = (m-1)!\ \mathrm{Pf}[c_{ij}]_{i,j=1,\ldots,2m}, \tag{3.8}$$

where $2m = N$ if N is even and $2m = N + 1$ if N is odd,

$$a_{ij} = \iint_{x \leqslant y} \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)[\theta_i(x)\theta_j(y) - \theta_j(x)\theta_i(y)] \tag{3.9}$$

for $i, j = 1, 2, \ldots, N$. When N is odd, we have in addition

$$a_{i,N+1} = -a_{N+1,i} = \int \mathrm{d}\mu(x)\theta_i(x) \tag{3.10}$$

for $i = 1, 2, \ldots, N$, and $a_{N+1,N+1} = 0$. Also

$$b_{ij} = \int \mathrm{d}\mu(x)[\theta_i(x)\tau_j(x) - \tau_i(x)\theta_j(x)] \tag{3.11}$$

for $i, j = 1, 2, \ldots, 2m$,

$$c_{ij} = b_{ij}, \qquad i, j = 1, 2, \ldots, 2m-1, \tag{3.12}$$

$$c_{i,2m} = -c_{2m,i} = \int \mathrm{d}\mu(x)\chi_i(x), \tag{3.13}$$

$$c_{2m,2m} = 0. \tag{3.14}$$

We therefore proceed in three steps.

Firstly we write H as a sum, $H = A + B$. This convenient breaking will be different according to whether $\alpha^2 < 1$ or $\alpha^2 > 1$ and has nothing to do with the previous writing of H as a sum of its real and imaginary parts.

Secondly we use equation (3.1) to integrate over the unitary matrix, diagonalising B. The result is some determinant containing the eigenvalues of H and of A, and the product of differences of these eigenvalues.

Finally we integrate over the eigenvalues of A using equations (3.3)–(3.14).

The treatment of the ensemble of Hermitian quaternion matrices will be parallel. Constants will be ignored in the intermediate steps; the final constant will be fixed by the normalisation condition (see appendices 2 and 3). The detailed working is given in the following sub-sections.

3.1. *Matrices from* GOE *to* GUE *and beyond*

We write H as a sum of two Hermitian matrices

$$H = A + B \tag{3.15}$$

and choose B so that its real and imaginary parts have the same variance. If $\alpha^2 < 1$, the real part of H has a larger variance than its imaginary part and we choose A to be real symmetric. If $\alpha^2 > 1$ it is the imginary part of H which has a larger variance and we choose A to be pure imaginary anti-symmetric. In either case we adjust the variance of A and the common variance of the real and imaginary parts of B in a proper way. Thus our choices are:

$$A^{\mathrm{T}} = A = A^*,$$

$$P_1(A) \propto \exp\{-\mathrm{Tr}\, A^2/[4v^2(1-\alpha^2)]\},$$

$$P_2(B) \propto \exp(-\mathrm{Tr}\, B^2/4v^2\alpha^2),$$

if $\alpha^2 < 1$, and

$$A^{\mathrm{T}} = -A = A^*,$$

$$P_1(A) \propto \exp\{-\mathrm{Tr}\, A^2/[4v^2(\alpha^2-1)]\},$$

$$P_2(B) \propto \exp(-\mathrm{Tr}\, B^2/4v^2),$$

if $\alpha^2 > 1$. We combine these equations as

$$A^{\mathrm{T}} = A^* = A \operatorname{sgn}(1-\alpha^2),$$

$$P_1(A) \propto \exp[-\operatorname{Tr} A^2/(4v^2|1-\alpha^2|)], \tag{3.16}$$

$$P_2(B) \propto \exp(-\operatorname{Tr} B^2/4v^2\gamma^2), \tag{3.17}$$

where $\gamma^2 = \min(1, \alpha^2)$. Though equations (3.16) and (3.17) look alike, they have an important difference: $\operatorname{Tr} B^2$ contains two sums of squares, those of the real and imaginary parts of B, whereas $\operatorname{Tr} A^2$ contains only one of them. Equation (2.2) is now written in the equivalent form

$$P(H) = \int P_1(A) P_2(H-A)\, \mathrm{d}A. \tag{3.18}$$

Let $x_1, \ldots, x_N$ be the eigenvalues of H

$$H = UxU^{\dagger}, \qquad U^{\dagger}U = 1, \tag{3.19}$$

so that (see Dyson 1962b, Mehta 1967 ch 3)

$$\mathrm{d}H \propto \Delta^2(x)\, \mathrm{d}U\, \mathrm{d}x, \tag{3.20}$$

and the joint probability density for the $x_1, \ldots, x_N$ is

$$p(x) \equiv p(x_1, \ldots, x_N) \propto \int P_1(A) P_2(H-A) \Delta^2(x)\, \mathrm{d}U\, \mathrm{d}A. \tag{3.21}$$

We consider separately the cases of A symetric and anti-symmetric.

When A is real symmetric, its real eigenvalues $a_1, \ldots, a_N$ are in general distinct and from (3.16), (3.17), (3.21) and (3.1) we have

$$p(x) \propto \int \exp\left(-\sum_1^N \frac{a_i^2}{4v^2(1-\alpha^2)}\right) \exp\left(-\frac{\operatorname{Tr}(UxU^{\dagger}-A)^2}{4v^2}\right) \Delta^2(x)\, \mathrm{d}U\, \mathrm{d}A \tag{3.22}$$

$$\propto \Delta(x) \int \exp\left(-\sum_1^N \frac{a_i^2}{4v^2(1-\alpha^2)}\right) \det\left[\exp\left(-\frac{(x_i-a_j)^2}{4v^2}\right)\right] \frac{1}{\Delta(a)}\, \mathrm{d}A. \tag{3.23}$$

As far as the dependence on the eigenvalues is concerned, one has (Dyson 1962b, Mehta 1967 ch 3)

$$\mathrm{d}A \propto |\Delta(a)|\, \mathrm{d}a_1 \ldots \mathrm{d}a_N, \tag{3.24}$$

so that

$$p(x) \propto \Delta(x) \int \exp\left(-\sum_1^N \frac{a_i^2}{4v^2(1-\alpha^2)}\right) \times \det\left[\exp\left(-\frac{(x_i-a_j)^2}{4v^2}\right)\right] \operatorname{sgn} \Delta(a)\, \mathrm{d}a_1 \ldots \mathrm{d}a_N. \tag{3.25}$$

Using equations (3.3), (3.6) and (3.9), we get

$$p(x) \propto \Delta(x) \exp\left(-\sum_1^N \frac{x_i^2}{4v^2}\right) \operatorname{Pf}[b_{ij}] \tag{3.26}$$

where

$$b_{ij}=\iint\limits_{-\infty\leq z_1\leq z_2\leq\infty} dz_1\,dz_2\Big[\exp\Big(-\frac{1}{4v^2\alpha^2(1-\alpha^2)}\{[z_1-(1-\alpha^2)x_i]^2+[z_2-(1-\alpha^2)x_j]^2\}\Big)$$

$$-\exp\Big(-\frac{1}{4v^2\alpha^2(1-\alpha^2)}\{[z_1-(1-\alpha^2)x_j]^2+[z_2-(1-\alpha^2)x_i]^2\}\Big)\Big], \tag{3.27}$$

for $i, j=1, 2, \ldots, N$; when the order N of the matrices is even. For N odd, we have one more row and column, for $i=1, \ldots, N$,

$$b_{i,N+1}=-b_{N+1,i}=\int_{-\infty}^{\infty} dz\,\exp\Big(-\frac{[z-(1-\alpha^2)x_i]^2}{4v^2\alpha^2(1-\alpha^2)}\Big)\propto 1. \tag{3.28}$$

On introducing the new variables $\mathscr{S}=(z_2+z_1)/\sqrt{2}$, $t=(z_2-z_1)/\sqrt{2}$, the integration on $\mathscr{S}$ in (3.27) can be performed. A little algebra then gives

$$b_{ij}=\text{constant erf}\Big[\Big(\frac{1-\alpha^2}{8v^2\alpha^2}\Big)^{1/2}(x_i-x_j)\Big] \qquad i, j=1, \ldots, N. \tag{3.29}$$

Equations (3.26), (3.28) and (3.29) give (2.7) when A is real symmetric.

When A is anti-symmetric pure imaginary, its eigenvalues are real and come in pairs $\pm a_i$; zero is necessarily an eigenvalue if the order N of A is odd. As far as the dependence on the eigenvalues is concerned (Dyson 1962b)

$$dA\propto\prod_{1\leq i<j\leq m}(a_i^2-a_j^2)^2\,da_1\ldots da_m, \tag{3.30}$$

$$\Delta(a)=2^m\prod_1^m a_i\prod_{1\leq i<j\leq m}(a_i^2-a_j^2)^2, \tag{3.31}$$

for $N=2m$ even; while

$$dA\propto\prod_1^m a_i^2\prod_{1\leq i<j\leq m}(a_i^2-a_j^2)^2, \tag{3.32}$$

$$\Delta(a)=2^m\prod_1^m a_i^3\prod_{1\leq i<j\leq m}(a_i^2-a_j^2)^2, \tag{3.33}$$

for $N=2m+1$ odd. (We take $a_1, \ldots, a_m$ as distinct positive numbers.) Thus from (3.16)–(3.18), (3.21), (3.1), (3.30) and (3.31) we get for $N=2m$ even

$$p(x)\propto\Delta(x)\exp\Big(-\sum_1^N\frac{x_i^2}{4v^2\alpha^2}\Big)\int_0^\infty\ldots\int_0^\infty\frac{da_1\ldots da_m}{a_1\ldots a_m},$$

$$\times\det\Big\{\exp\Big[-\frac{\alpha^2}{4v^2(\alpha^2-1)}\Big(a_j-\frac{\alpha^2-1}{\alpha^2}x_i\Big)^2\Big],$$

$$\exp\Big[-\frac{\alpha^2}{4v^2(\alpha^2-1)}\Big(a_j+\frac{\alpha^2-1}{\alpha^2}x_i\Big)^2\Big]\Big\},$$

$$i=1, \ldots, N; \qquad j=1, \ldots, m. \tag{3.34}$$

For $N = 2m+1$ odd, instead of (3.30) and (3.31) we use (3.32) and (3.33). The result is again equation (3.34) in which the determinant contains one more column

$$\exp\left[-\frac{\alpha^2}{4v^2(\alpha^2-1)}\left(\frac{\alpha^2-1}{\alpha^2}x_i\right)^2\right] = \exp\left(-\frac{\alpha^2-1}{4v^2\alpha^2}x_i^2\right). \tag{3.35}$$

For N even, we use equations (3.4), (3.7) and (3.11), to get from (3.34)

$$p(x) \propto \Delta(x)\exp\left(-\sum_1^N \frac{x_i^2}{4v^2\alpha^2}\right)\mathrm{Pf}[b_{ij}] \tag{3.36}$$

with

$$b_{ij} = \int_0^\infty \frac{\mathrm{d}z}{z}\left[\!\left[\exp\left\{-\frac{\alpha^2}{4v^2(\alpha^2-1)}\left[\left(z-\frac{\alpha^2-1}{\alpha^2}x_i\right)^2+\left(z+\frac{\alpha^2-1}{\alpha^2}x_j\right)^2\right]\right\}\right.\right.$$
$$\left.\left. -\exp\left\{-\frac{\alpha^2}{4v^2(\alpha^2-1)}\left[\left(z+\frac{\alpha^2-1}{\alpha^2}x_i\right)^2+\left(z-\frac{\alpha^2-1}{\alpha^2}x_j\right)^2\right]\right\}\right]\!\right]. \tag{3.37}$$

For N odd, we use (3.5), (3.8), (3.12) and (3.13); the result is again (3.36) in which the Pfaffian contains one more row and a column

$$b_{i,N+1} = -b_{N+1,i} = \exp\left(-\frac{\alpha^2-1}{4v^2\alpha^2}x_i^2\right). \tag{3.38}$$

A little algebra gives for (3.37)

$$b_{ij} \propto \exp\left(-\frac{\alpha^2-1}{4v^2\alpha^2}(x_i^2+x_j^2)\right)\int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-t^2}\sinh\left[\left(\frac{\alpha^2-1}{2v^2\alpha^2}\right)^{1/2}(x_i-x_j)t\right]$$
$$\propto \exp\left(-\frac{\alpha^2-1}{4v^2\alpha^2}(x_i^2+x_j^2)\right)\mathrm{erf}\left[\left(\frac{\alpha^2-1}{8v^2\alpha^2}\right)^{1/2}(x_i-x_j)\right]. \tag{3.39}$$

Equations (3.36), (3.39) and (3.38) give equation (2.7) for anti-symmetric A.

Equation (2.7) is therefore established in all cases except for the overall constant.

3.2. *Matrices from* GSE *to* GUE *and beyond*

We write the $N \times N$ quaternion matrix $\boldsymbol{H}$ as a sum

$$\boldsymbol{H} = \boldsymbol{A} + \boldsymbol{B}, \qquad \bar{\boldsymbol{A}} = \boldsymbol{A}\,\mathrm{sgn}(1-\alpha^2). \tag{3.40}$$

The quaternion matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are Hermitian; the self-dual and anti-self-dual parts of $\boldsymbol{B}$ have the same variance; $\boldsymbol{A}$ is self-dual quaternion real if $\alpha^2 < 1$, and anti-self-dual quaternion pure imaginary if $\alpha^2 > 1$. Due to equation (3.2) equation (2.45) is equivalent to

$$P_1(\boldsymbol{A}) \propto \exp\left[-\mathrm{Tr}\,\boldsymbol{A}^2/(4v^2|1-\alpha^2|)\right], \tag{3.41}$$

$$P_2(\boldsymbol{B}) \propto \exp\left(-\mathrm{Tr}\,\boldsymbol{B}^2/4v^2\gamma^2\right), \tag{3.42}$$

where as before $\gamma^2 = \min(1, \alpha^2)$. Let

$$\boldsymbol{H} = UxU^\dagger, \qquad UU^\dagger = 1, \tag{3.43}$$

where $\boldsymbol{H}$, x and U are $2N\times 2N$ ordinary matrices, x is diagonal, the diagonal elements $x_1,\ldots,x_{2N}$ of x are the eigenvalues of $\boldsymbol{H}$. As before (Dyson 1962b, Mehta 1967 ch 3)

$$\mathrm{d}\boldsymbol{H}\propto\Delta^2(x_1,\ldots,x_{2N})\,\mathrm{d}x\,\mathrm{d}U, \tag{3.44}$$

$$\mathrm{d}x\equiv\mathrm{d}x_1\ldots\mathrm{d}x_{2N}. \tag{3.45}$$

We consider separately, as before, the cases of $\boldsymbol{A}$ self-dual and $\boldsymbol{A}$ anti-self-dual.

When $\boldsymbol{A}$ is self-dual, its eigenvalues are real numbers $a_1,\ldots,a_N$ each repeated twice. Since the right-hand side of equation (3.1) is now of the form 0/0, we cannot use it directly, but have to take limits when the eigenvalues of $\boldsymbol{A}$ become equal in pairs. The calculation gives

$$\int\exp[-\mathrm{Tr}(\boldsymbol{A}-UxU^{\dagger})^2/4v^2\alpha^2]\,\mathrm{d}U\propto(\Delta(x)\Delta^4(a))^{-1}$$
$$\times\det\left[\exp\left(-\frac{(a_i-x_j)^2}{4v^2\alpha^2}\right),\qquad(a_i-x_j)\exp\left(-\frac{(a_i-x_j)^2}{4v^2\alpha^2}\right)\right], \tag{3.46}$$

$i=1,\ldots,N$; $j=1,\ldots,2N$, where $\Delta(x)\equiv\Delta(x_1,\ldots,x_{2N})$ and $\Delta(a)\equiv\Delta(a_1,\ldots,a_N)$, equation (2.8).

Now as far as the dependence on the eigenvalues is concerned (Dyson 1962b, Mehta 1967 ch 3)

$$\mathrm{d}\boldsymbol{A}\propto\Delta^4(a)\,\mathrm{d}a_1\ldots\mathrm{d}a_N\equiv\Delta^4(a)\,\mathrm{d}a, \tag{3.47}$$

so that

$$\boldsymbol{p}(x)\equiv\boldsymbol{p}(x_1,\ldots,x_{2N})\propto\Delta(x)\int\mathrm{d}a\,\exp\left(-\sum_1^N\frac{a_i^2}{2v^2(1-\alpha^2)}\right)$$
$$\times\det\left[\exp\left(-\frac{(a_i-x_j)^2}{4v^2\alpha^2}\right),\qquad(a_i-x_j)\exp\left(-\frac{(a_i-x_j)^2}{4v^2\alpha^2}\right)\right]. \tag{3.48}$$

Using equations (3.4), (3.7) and (3.11) one gets

$$\boldsymbol{p}(x)\propto\Delta(x)\exp\left(-\sum_1^{2N}\frac{x_i^2}{4v^2}\right)\mathrm{Pf}[h_{ij}], \tag{3.49}$$

where

$$h_{ij}=(x_i-x_j)\exp\left(\frac{x_i^2+x_j^2}{4v^2}\right)\int_{-\infty}^{\infty}\mathrm{d}z\,\exp\left(-\frac{z^2}{2v^2(1-\alpha^2)}\right)\exp\left(-\frac{(z-x_i)^2}{4v^2\alpha^2}-\frac{(z-x_j)^2}{4v^2\alpha^2}\right). \tag{3.50}$$

A little algebra now gives

$$h_{ij}\propto(x_i-x_j)\exp\left(-\frac{1-\alpha^2}{8v^2\alpha^2}(x_i-x_j)^2\right). \tag{3.51}$$

From (3.49) and (3.51) we get equation (2.49) for the case $\alpha^2<1$.

When A is anti-self-dual, its eigenvalues are the real numbers $\pm a_1, \pm a_2, \ldots, \pm a_N$, and from equation (3.1)

$$\int \exp\left(-\frac{\operatorname{Tr}(\boldsymbol{A}-UxU^{+})^2}{4v^2}\right) \mathrm{d}U \propto [\Delta(x)\Delta(\pm a_1, \ldots, \pm a_N)]^{-1} \times \det\left[\exp\left(-\frac{(a_i-x_j)^2}{4v^2}\right), \quad \exp\left(-\frac{(a_i+xj)^2}{4v^2}\right)\right]. \tag{3.52}$$

However,

$$\Delta(\pm a_1, \ldots, \pm a_N) = 2^N \prod_1^N a_i[\Delta(a_1^2, \ldots, a_N^2)]^2, \tag{3.53}$$

and as far as depeendence on the eigenvalues is concerned (Dyson 1962b)

$$\mathrm{d}\boldsymbol{A} \propto \prod_1^N a_i^2 \quad \left[\Delta(a_1^2, \ldots, a_N^2)\right]^2, \tag{3.54}$$

so that

$$p(x) \propto \Delta(x) \int_0^\infty \ldots \int_0^\infty \mathrm{d}a_1 \ldots \mathrm{d}a_N \prod_1^N a_i \exp\left(-\sum_1^N \frac{a_i^2}{2v^2(\alpha^2-1)}\right) \times \det\left[\exp\left(\frac{(a_i-x_j)}{4v^2}\right), \quad \exp\left(-\frac{(a_i+x_j)^2}{4v^2}\right)\right] \propto \Delta(x) \exp\left(-\sum_1^{2N} \frac{x_i^2}{4v^2}\right) \operatorname{Pf}[h_{ij}], \tag{3.55}$$

where in the last step we have used equations (3.4), (3.7), (3.11), and

$$h_{ij} = \exp\left(\frac{x_i^2+x_j^2}{4v^2}\right) \int_0^\infty \mathrm{d}z\, z \exp\left(-\frac{z^2}{2v^2(\alpha^2-1)}\right) \times \left[\exp\left(-\frac{(z-x_i)^2}{4v^2} - \frac{(z+x_j)^2}{4v^2}\right) - \exp\left(-\frac{(z+x_i)^2}{4v^2} - \frac{(z-x_j)^2}{4v^2}\right)\right]. \tag{3.56}$$

A little algebra now gives

$$h_{ij} \propto (x_i - x_j) \exp\left(-\frac{\alpha^2-1}{8v^2\alpha^2}(x_i-x_j)^2\right). \tag{3.57}$$

Equations (3.55) and (3.57) give (2.49) for $\alpha^2 > 1$.

Equation (2.49) is therefore established for all values of α^2, except for the overall constant.

4. Correlation and cluster functions

To derive the correlation functions we shall use the theory of quaternion matrices (Dyson 1970, 1972, Mehta 1971, 1977 ch 8).

Definition. Let Q be an $N \times N$ quaternion self-dual matrix. We define $T\det Q$ to be the scalar

$$T \det Q = \sum (-1)^P (q_{i_1 i_2} q_{i_2 i_3} \ldots q_{i_r i_1})_0 (q_{j_1 j_2} q_{j_2 j_3} \ldots q_{j_s j_1})_0 \ldots \tag{4.1}$$

where the sum is taken over all permutations P consisting of exclusive cycles $(i_1 i_2 \ldots i_r)$, $(j_1 j_2 \ldots j_s) \ldots$, and the subscript zero denotes the scalar part.

In particular we need the following two theorems (Dyson 1970, 1972, Mehta 1971, 1977 ch 8).

Theorem 4.1. Let Q_N be an $N \times N$ quaternion self-dual matrix with elements $q_{ij} = q(x_i, x_j) = \bar{q}_{ji}$, satisfying the conditions

$$\int_{-\infty}^{\infty} q(x, y) q(y, z) \, \mathrm{d}y = q(x, z) + \tau q(x, z) - q(x, z)\tau, \tag{4.2}$$

with τ a constant quaternion. Then

$$\int_{-\infty}^{\infty} T\det Q_N \, \mathrm{d}x_N = (c - N + 1) T\det Q_{N-1}, \tag{4.3}$$

where c is the constant scalar

$$c = \int_{-\infty}^{\infty} q(x, x) \, \mathrm{d}x \tag{4.4}$$

and Q_{N-1} is the $(N-1) \times (N-1)$ quaternion self-dual matrix obtained by removing from Q_N the row and column containing the variable x_N.

Theorem 4.2. Let Q be an $N \times N$ self-dual quaternion matrix. Let us denote by $M(Q)$ the $2N \times 2N$ matrix obtained from Q when its quaternion elements are replaced by their 2×2 matrix representatives, equation (2.37). Then

$$\det M(Q) = (T\det Q)^2. \tag{4.5}$$

Let us consider the first ensemble. Using theorem 4.1 several times one can get the correlation function

$$R_n(x_1, \ldots, x_n) = \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} p(x_1, \ldots, x_N) \, \mathrm{d}x_{n+1} \ldots \mathrm{d}x_N \tag{4.6}$$

as a Tdet of an $n \times n$ quaternion matrix; the derivation relies on the fact that the $p(x)$ of equation (2.7) can be written as

$$p(x_1, \ldots, x_N) = (1/N!) T\det[\Phi(x_i, x_j)]_{i,j=1,\ldots,N} \tag{4.7}$$

where $\Phi(x, y)$ (equation (2.14)) satisfies the equality

$$\int_{-\infty}^{\infty} \Phi(x, y) \Phi(y, z) \, \mathrm{d}y = \Phi(x, z) + \tau\Phi(x, z) - \Phi(x, z)\tau, \tag{4.8}$$

in which τ is a constant quaternion. Equation (2.12) then follows from theorem 4.2. The proof of equation (4.7) is given in appendix 4 and that of (4.8) in appendix 5.

Similarly one can get the correlation functions for the other ensemble, since

$$\boldsymbol{p}(x_1, \ldots, x_{2N}) = [(2N)!]^{-1} T\det[\boldsymbol{\Phi}(x_i, x_j)]_{i,j=1,\ldots,2N}, \tag{4.9}$$

where $\boldsymbol{p}$ is now given by (2.49) and $\boldsymbol{\Phi}$ by (2.52), and since this $\boldsymbol{\Phi}$ satisfies equation (4.8).

The proof of equation (4.9) is given in appendix 4 and that of (4.8) with Φ replaced by $\boldsymbol{\Phi}$ is given in appendix 5.

Equation (2.13) is obtained from the observation that (Dyson 1970) the cluster function T_n is the cumulant of the correlation function R_n and the expression on the right-hand side of (2.13) is the cumulant of the $T\det[\Phi(x_i, x_j)]_{i,j=1,2,\ldots,n}$.

To get the limits for large N of the correlation and cluster functions it suffices then to take the limits of $\Phi(x, y)$ and $\boldsymbol{\Phi}(x, y)$. The limits of the functions $S_N(x, y)$, $D_N(x, y)$, $I_N(x, y)$ and $J_N(x, y)$ were derived by Pandey and Mehta (1983). Those of $\boldsymbol{S}_N$, $\boldsymbol{D}_N$ and $\boldsymbol{I}_N$ are obtained from those of S_N, D_N and I_N by changing λ^2 to $-\lambda^2$ and replacing N by $2N$. For $\boldsymbol{g}$ we have

$$[\boldsymbol{R}_1(x)]^{-2}\boldsymbol{g}(x, y) \approx -(r/8\lambda^3\sqrt{\pi})\exp(-r^2/8\lambda^2) \approx -\pi^{-1}\int_0^\infty k \sin kr \exp(-2\lambda^2k^2)\,\mathrm{d}k, \tag{4.10}$$

from which we get the limit of $\boldsymbol{K}_N$.

This completes the proof of the statements listed in § 2. Some technical details are collected in the appendices.

Acknowledgment

One of us (AP) wants to thank Dr S B Khadkikar for a helpful remark.

Appendix 1. Some useful relations

In this appendix we collect some recurrence relations and identities often used in the text and in later appendices.

A1.1

The harmonic oscillator functions

$$\varphi_j(x) = (2^j j!\sqrt{\pi})^{-1/2}\exp(\tfrac{1}{2}x^2)\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^j \exp(-x^2), \tag{2.26}$$

obey the orthonormality

$$\int_{-\infty}^{\infty} \varphi_i(x)\varphi_j(x)\,\mathrm{d}x = \delta_{ij} \tag{A1.1}$$

and the recurrence relations

$$\sqrt{2}\varphi_j'(x) = \sqrt{j}\,\varphi_{j-1}(x) - \sqrt{j+1}\,\varphi_{j+1}(x), \tag{A1.2}$$

$$\sqrt{2}\,x\varphi_j(x) = \sqrt{j}\,\varphi_{j-1}(x) + \sqrt{j+1}\,\varphi_{j+1}(x). \tag{A1.3}$$

These equalities follow from those for Hermite polynomials (Bateman 1953, Szegö 1939) and will not be proved here.

A1.2

The functions $\psi_j(x)$ and $A_j(x)$ defined by (2.27) and (2.28) satisfy the equations

$$\sqrt{2}\,\psi_j(x)=[(1+\alpha^2)/(1-\alpha^2)]^j[\sqrt{j}\,(1-\alpha^2)\varphi_{j-1}(x)-\sqrt{j+1}\,(1+\alpha^2)\varphi_{j+1}(x)], \qquad (A1.4)$$

$$\sqrt{2}\,\varphi_j(x)=[(1+\alpha^2)/(1-\alpha^2)]^j[\sqrt{j}(1-\alpha^2)A_{j-1}(x)-\sqrt{j+1}\,(1+\alpha^2)A_{j+1}(x)]. \qquad (A1.5)$$

These relations can be easily derived from (A1.2) and (A1.3).

A1.3

The orthonormality relations involving $\psi_j(x)$, $A_j(x)$ and $\varphi_j(x)$ are

$$\int_{-\infty}^{\infty}\varphi_i(x)\varphi_j(x)\,\mathrm{d}x=\delta_{ij}, \qquad (A1.1)$$

$$\int_{-\infty}^{\infty}\psi_i(x)A_j(x)\,\mathrm{d}x=-\delta_{ij}, \qquad (A1.6)$$

$$\int_{-\infty}^{\infty}\psi_i(x)\varphi_j(x)\,\mathrm{d}x=\int_{-\infty}^{\infty}A_i(x)\varphi_j(x)\,\mathrm{d}x=0, \qquad i+j \text{ even}. \qquad (A1.7)$$

The second is obtained by partial integration and the third by parity argument.

A1.4

The recurrence relations

$$\begin{aligned}\psi_{2j}(x)A_{2j}(y)&\\ &=-\varphi_{2j+1}(x)\varphi_{2j+1}(y)+(1-\alpha^2)\{[(1+\alpha^2)/(1-\alpha^2)]^{2j}\sqrt{j}\,\varphi_{2j-1}(x)A_{2j}(y)\\ &\quad-[(1+\alpha^2)/(1-\alpha^2)]^{2j+2}\sqrt{j+1}\,\varphi_{2j+1}(x)A_{2j+2}(y)\}, \qquad (A1.8)\end{aligned}$$

$$\begin{aligned}\psi_{2j+1}(x)A_{2j+1}(y)&\\ &=-\varphi_{2j}(x)\varphi_{2j}(y)+(1-\alpha^2)\{[(1+\alpha^2)/(1-\alpha^2)]^{2j}\sqrt{j}\,\varphi_{2j}(x)A_{2j-1}(y)\\ &\quad-[(1+\alpha^2)/(1-\alpha^2)]^{2j+2}\sqrt{j+1}\,\varphi_{2j+2}(x)A_{2j+1}(y)\}, \qquad (A1.9)\end{aligned}$$

$$\begin{aligned}\varphi_{2j}(x)A_{2j}(y)+A_{2j+1}(x)\varphi_{2j+1}(y)&\\ &=(1-\alpha^2)\{[(1+\alpha^2)/(1-\alpha^2)]^{2j}\sqrt{j}\,A_{2j-1}(x)A_{2j}(y)\\ &\quad-[(1+\alpha^2)/(1-\alpha^2)]^{2j+2}\sqrt{j+1}\,A_{2j+1}(x)A_{2j+2}(y)\}, \qquad (A1.10)\end{aligned}$$

$$\begin{aligned}\psi_{2j}(x)\varphi_{2j}(y)+\varphi_{2j+1}(x)\psi_{2j+1}(y)&\\ &=(1-\alpha^2)\{[(1+\alpha^2)/(1-\alpha^2)]^{2j}\sqrt{j}\,\varphi_{2j-1}(x)\varphi_{2j}(y)\\ &\quad-[(1+\alpha^2)/(1-\alpha^2)]^{2j+2}\sqrt{j+1}\,\varphi_{2j+1}(x)\varphi_{2j+2}(y)\}, \qquad (A1.11)\end{aligned}$$

can be derived from (A1.4) and (A1.5). These relations are useful to verify the equivalence of expressions (2.16) and (2.17) of (2.18) and (2.19) and of (2.21) and (2.22).

A1.5

We need the integral

$$\int_{-\infty}^{\infty} \varphi_{2j}(y)\exp(-\tfrac{1}{2}\alpha^2y^2)\,\mathrm{d}y$$
$$=[2\pi/(1+\alpha^2)]^{1/2}\frac{(2j)!}{j!}(2^{2j}(2j)!\sqrt{\pi})^{-1/2}[(1-\alpha^2)/(1+\alpha^2)]^j. \qquad \text{(A1.12)}$$

To prove this equation we use the generating function

$$\exp(-y^2+2yz-z^2)=\sum_{j=0}^{\infty}\frac{z^j}{j!}\left(-\frac{\mathrm{d}}{\mathrm{d}y}\right)^j\exp(-y^2)$$
$$=\sum_{j=0}^{\infty}\frac{z^j}{j!}\exp(-\tfrac{1}{2}y^2)(2^jj!\sqrt{\pi})^{1/2}\varphi_j(y) \qquad \text{(A1.13)}$$

to write

$$\sum_{j=0}^{\infty}\frac{z^j}{j!}(2^jj!\sqrt{\pi})^{1/2}\int_{-\infty}^{\infty}\varphi_j(y)\exp(-\tfrac{1}{2}\alpha^2y^2)\,\mathrm{d}y$$
$$=\int_{-\infty}^{\infty}\exp[-\tfrac{1}{2}(1+\alpha^2)y^2+2yz-z^2]\,\mathrm{d}y$$
$$=\exp\left[z^2\left(\frac{1-\alpha^2}{1+\alpha^2}\right)\right]\int_{-\infty}^{\infty}\exp\left[-\frac{1+\alpha^2}{2}\left(y-\frac{2}{1+\alpha^2}z\right)^2\right]\mathrm{d}y$$
$$=\left(\frac{2\pi}{1+\alpha^2}\right)^{1/2}\sum_{j=0}^{\infty}\frac{z^{2j}}{j!}\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^j.$$

Equating coefficients of z^{2j} on both sides we get (A1.12).

A1.6

We need the convolution integrals

$$\int_{-\infty}^{\infty} g(x,y)\psi_j(y)\,\mathrm{d}y=\varphi_j(x), \qquad \text{(A1.14)}$$

$$\int_{-\infty}^{\infty} g(x,y)\varphi_j(y)\,\mathrm{d}y=A_j(x), \qquad \text{(A1.15)}$$

with $g(x, y)$ given by equation (2.23). By partial integration (A1.14) is equivalent to

$$\int_{-\infty}^{\infty}\exp\left[-\tfrac{1}{2}\alpha^2(x^2+y^2)-\left(\frac{1-\alpha^4}{4\alpha^2}\right)(x-y)^2\right]\varphi_j(y)\,\mathrm{d}y=\frac{2\alpha\sqrt{\pi}}{1+\alpha^2}\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^j\varphi_j(x). \qquad \text{(A1.16)}$$

To prove this we expand both sides of the identity

$$\int_{-\infty}^{\infty}\exp\left[2yz-z^2-\tfrac{1}{2}(1+\alpha^2)y^2-\left(\frac{1-\alpha^4}{4\alpha^2}\right)(x-y)^2\right]\mathrm{d}y$$
$$=\frac{2\alpha\sqrt{\pi}}{1+\alpha^2}\exp\left[-\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^2z^2+2\left(\frac{1-\alpha^2}{1+\alpha^2}\right)zx-\tfrac{1}{2}(1-\alpha^2)x^2\right] \qquad \text{(A1.17)}$$

in powers of z and use the generating function (A1.13).

Equation (A1.15) can be rewritten as

$$\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{j+\frac{1}{2}}\int_{-\infty}^{\infty} f(x-y)\exp(-\tfrac{1}{2}\alpha^2y^2)\varphi_j(y)\,dy=\int_{-\infty}^{\infty}\exp(\tfrac{1}{2}\alpha^2y^2)\varphi_j(y)\,\mathrm{sgn}(x-y)\,dy.$$

This last equation is true, since the derivatives of both sides are equal, equation (A1.16), and the equality holds at one point $x=\infty$. In fact at $x=\infty$, $f(x,y)=\mathrm{sgn}(x-y)=1$, and we have to make sure that

$$\left(\frac{1+\alpha^2}{1-\alpha^2}\right)^{j+1/2}\int_{-\infty}^{\infty}\exp(-\tfrac{1}{2}\alpha^2y^2)\varphi_j(y)\,dy=\int_{-\infty}^{\infty}\exp(\tfrac{1}{2}\alpha^2y^2)\varphi_j(y)\,dy. \qquad (A1.18)$$

For j odd, both sides are zero and the equality is evident. For j even, changing the sign of α^2 in (A1.12) and taking ratios we get (A1.18).

A1.7 Recurrence and orthonormality relations between $\boldsymbol{\psi}_j(x)$, $\boldsymbol{A}_j(x)$ and $\varphi_j(x)$.

The functions $\boldsymbol{\psi}_j(x)$ and $\boldsymbol{A}_j(x)$, equations (2.62) and (2.63), are obtained from $\psi_j(x)$ and $A_j(x)$, equations (2.27) and (2.28), by changing the sign of α^2. So from (A1.4)–(A1.7) we get

$$\sqrt{2}\,\boldsymbol{\psi}_j(x)=[(1-\alpha^2)/(1+\alpha^2)]^j[\sqrt{j}\,(1+\alpha^2)\varphi_{j-1}(x)-\sqrt{j+1}\,(1-\alpha^2)\varphi_{j+1}(x)], \qquad (A1.19)$$

$$\sqrt{2}\,\varphi_j(x)=[(1-\alpha^2)/(1+\alpha^2)]^j[\sqrt{j}\,(1+\alpha^2)\boldsymbol{A}_{j-1}(x)-\sqrt{j+1}\,(1-\alpha^2)\boldsymbol{A}_{j+1}(x)], \qquad (A1.20)$$

$$\int_{-\infty}^{\infty}\boldsymbol{\psi}_i(x)\boldsymbol{A}_j(x)\,dx=-\delta_{ij}, \qquad (A1.21)$$

$$\int_{-\infty}^{\infty}\boldsymbol{\psi}_i(x)\varphi_j(x)\,dx=\int_{-\infty}^{\infty}\boldsymbol{A}_i(x)\varphi_j(x)\,dx=0, \qquad i+j \text{ even}. \qquad (A1.22)$$

A1.8

We also need the convolution integrals

$$\int_{-\infty}^{\infty}\boldsymbol{g}(x,y)\varphi_j(y)\,dy=\boldsymbol{\psi}_j(x), \qquad (A1.23)$$

$$\int_{-\infty}^{\infty}\boldsymbol{g}(x,y)\boldsymbol{A}_j(y)\,dy=\varphi_j(x). \qquad (A1.24)$$

Equation (A1.23) can be established from the definitions of $\boldsymbol{g}(x,y)$, $\boldsymbol{\psi}_j(x)$, equations (2.60), (2.62), and using equations (A1.16), (A1.2) and (A1.3). For equation (A1.24) we have from the definitions of $\boldsymbol{g}(x,y)$ and $\boldsymbol{A}_j(x)$, equations (2.60), (2.63),

$$\int_{-\infty}^{\infty}\boldsymbol{g}(x,y)\boldsymbol{A}_j(y)\,dy=-\frac{(1+\alpha^2)(1-\alpha^4)}{4\alpha^3\sqrt{\pi}}\left(\frac{1+\alpha^2}{1-\alpha^2}\right)^j\iint_{-\infty}^{\infty}dy\,dz$$
$$\times\,\varepsilon(y-z)\varphi_j(z)\exp[-\tfrac{1}{2}\alpha^2(x^2+z^2)](x-y)\exp\left(-\frac{1-\alpha^4}{4\alpha^2}(x-y)^2\right).$$

Changing the variable y to $t = x - y$, one can integrate over t

$$\frac{1-\alpha^4}{-2\alpha^2}\int_{-\infty}^{\infty} \varepsilon(x-z-t)t \exp\left(-\frac{1-\alpha^4}{4\alpha^2}t^2\right) \mathrm{d}t = \exp\left(-\frac{1-\alpha^4}{4\alpha^2}(x-z)^2\right).$$

Now using equation (A1.16) one gets (A1.24).

Appendix 2. A normalisation integral

The constant in (2.11) being fixed by the condition

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} p(x_1, \ldots, x_N)\, \mathrm{d}x_1 \ldots \mathrm{d}x_N = 1 \tag{A2.1}$$

with $p(x_1, \ldots, x_N)$ given by (2.7), we shall evaluate the integral

$$C_N^{-1} = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_N \exp\left(-\frac{1+\alpha^2}{2}\sum x_i^2\right)\Delta(x)\, \mathrm{Pf}[F_{ij}], \tag{A2.2}$$

with

$$\Delta(x) = \prod_{i<j} (x_i - x_j), \tag{2.8}$$

and F_{ij} defined by equations (2.9) and (2.9a).

The Pfaffian and $\Delta(x)$ are alternating functions of the variables $x_1, \ldots, x_N$. The $\Delta(x)$ can be written as a determinant

$$\exp\left(-\tfrac{1}{2}\sum x_i^2\right)\Delta(x) = \prod_0^{N-1} (2^{-j} j!\sqrt{\pi})^{1/2} \det[\varphi_{i-1}(x_j)]_{i,j=1,\ldots,N,} \tag{A2.3}$$

where $\varphi_i(x)$ is the normalised 'oscillator function' given by (2.26). The number of terms in the Pfaffian is $(2m)!/(2^m m!)$ with $N = 2m$ or $N = 2m - 1$.

It is convenient to discuss separately the cases N even and N odd. Let us first take the case $N = 2m$ even. From the symmetry and what we said above, one has

$$C_N^{-1} = \prod_0^{N-1} (2^{-j} j!\sqrt{\pi})^{1/2} \int_{-\infty}^{\infty} \ldots \int \mathrm{d}x_1 \ldots \mathrm{d}x_N \exp\left(-\tfrac{1}{2}\alpha^2 \sum x_i^2\right)$$

$$\times \mathrm{Pf}[f(x_i - x_j)] \det[\varphi_{i-1}(x_j)] \tag{A2.4}$$

$$= \prod_0^{N-1} (2^{-j} j!\sqrt{\pi})^{1/2} \frac{(2m)!}{2^m m!} \int_{-\infty}^{\infty} \ldots \int \mathrm{d}x_1 \ldots \mathrm{d}x_N \exp\left(-\tfrac{1}{2}\alpha^2 \sum x_i^2\right)$$

$$\times \prod_{j=1}^{m} f(x_{2j} - x_{2j-1}) \det[\varphi_{i-1}(x_j)]. \tag{A2.5}$$

For the last integral, we have from the theory of Pfaffians (Mehta 1977 appendix A.7)

$$C_N^{-1} \prod_0^{N-1} (2^{-j} j!\sqrt{\pi})^{-1/2} \left(\frac{(2m)!}{2^m m!}\right)^{-1} = m!\,2^m\, \mathrm{Pf}[a_{ij}]_{i,j=0,1,\ldots,2m-1} \tag{A2.6}$$

where for $i, j = 0, 1, \ldots, 2m-1$,

$$a_{ij} = \tfrac{1}{2} \iint\limits_{-\infty}^{\infty} \exp[\tfrac{1}{2}\alpha^2(x^2+y^2)] f(y-x)[\varphi_i(x)\varphi_j(y) - \varphi_j(x)\varphi_i(y)]\, dx\, dy$$

$$= \iint\limits_{-\infty}^{\infty} \exp[-\tfrac{1}{2}\alpha^2(x^2+y^2)] f(y-x)\varphi_i(x)\varphi_j(y)\, dx\, dy. \tag{A2.7}$$

From parity one sees that $a_{ij} = 0$ if $i+j$ is even. Thus the Pfaffian reduces to a determinant

$$\mathrm{Pf}[a_{ij}]_{i,j=0,1,\ldots,2m-1} = \det[a_{2i,2j+1}]_{i,j=0,1,\ldots,m-1}. \tag{A2.8}$$

The determinant is not changed if we add to any row (column) a constant multiple of another row (column). To choose convenient multiples, we observe that using the identity (A1.4)

$$\sqrt{2}\frac{d}{dy}[\varphi_j(y)\exp(-\tfrac{1}{2}\alpha^2 y^2)]$$
$$= \exp(-\tfrac{1}{2}\alpha^2 y^2)[\sqrt{j}(1-\alpha^2)\varphi_{j-1}(y) - \sqrt{j+1}\,(1+\alpha^2)\varphi_{j+1}(y)], \tag{A2.9}$$

an integration by parts in equation (A2.7) gives

$$(1/\sqrt{2})[\sqrt{j}\,(1-\alpha^2)a_{i,j-1} - \sqrt{j+1}\,(1+\alpha^2)a_{i,j+1}]$$
$$= -\frac{2}{\sqrt{\pi}}\left(\frac{1-\alpha^4}{4\alpha^2}\right)^{1/2} \iint\limits_{-\infty}^{\infty} \exp\biggl(-\tfrac{1}{2}\alpha^2(x^2+y^2)$$
$$-\frac{1-\alpha^4}{4\alpha^2}(x-y)^2\biggr)\varphi_i(x)\varphi_j(y)\, dx\, dy$$
$$= -2[(1-\alpha^2)/(1+\alpha^2)]^{j+1/2}\delta_{ij}. \tag{A2.10}$$

In the last step we made use of equations (A1.16) and (A1.1). Thus

$$a_{2i,2j+1} - \left(\frac{2j}{2j+1}\right)^{1/2}\left(\frac{1-\alpha^2}{1+\alpha^2}\right)a_{2i,2j-1} = \frac{2}{1+\alpha^2}\left(\frac{2}{2j+1}\right)^{1/2}\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{2j+1/2}\delta_{ij} = b_{ij}, \text{ say,} \tag{A2.11}$$

and

$$\det[a_{2i,2j+1}] = \det[b_{ij}] = \prod_{j=0}^{m-1}\left[\frac{2}{1+\alpha^2}\left(\frac{2}{2j+1}\right)^{1/2}\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{2j+1/2}\right]. \tag{A2.12}$$

From (A2.6), (A2.8) and (A2.12) we finally have for $N = 2m$,

$$C_N^{-1} = \prod_0^{N-1}(2^{-j}j!\sqrt{\pi})^{1/2} N! \prod_0^{m-1}\left[\frac{2}{1+\alpha^2}\left(\frac{2}{2j+1}\right)^{1/2}\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{2j+1/2}\right] \tag{A2.13}$$

which is equation (2.11), since

$$\prod_0^{N-1} (\sqrt{\pi}\, j!)^{1/2} = \prod_0^{m-1} (\pi(2j)!(2j+1)!)^{1/2} = \prod_0^{m-1} [(2j+1)^{1/2}\pi^{1/2}\Gamma(2j+1)]$$

$$= \prod_0^{m-1} \left[(2j+1)^{1/2}2^{2j}\Gamma(j+\tfrac{1}{2})\Gamma(j+1)\right]$$

$$= \prod_0^{m-1} (2j+1)^{1/2} 2^{m(m-1)} \prod_1^{2m} \left(\frac{2}{j}\Gamma(1+\tfrac{1}{2}j)\right). \tag{A2.14}$$

When $N = 2m-1$ is odd, we have similarly

$$C_N^{-1} \prod_0^{N-1} (2^{-j} j! \sqrt{\pi})^{-1/2}$$

$$= \frac{(2m)!}{2^m m!} \int_{-\infty}^{\infty} \cdots \int \mathrm{d}x_1 \ldots \mathrm{d}x_N \exp\left(-\tfrac{1}{2}\alpha^2 \sum x_i^2\right) \prod_{j=1}^{m-1} f(x_{2j} - x_{2j-1}) \det[\varphi_{i-1}(x_j)]$$

$$= \frac{(2m)!}{2^m m!} 2^{m-1}(m-1)!\, \mathrm{Pf}[a_{ij}]_{i,j=0,1\ldots,2m-1}, \tag{A2.15}$$

where now for $i, j = 0, 1, \ldots, 2m-2$, a_{ij} is given by (A2.7), and

$$a_{i,2m-1} = -a_{2m-1,i} = \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}\alpha^2 x^2)\phi_i(x)\, \mathrm{d}x, \qquad i = 0, 1, \ldots, 2m-2, \tag{A2.16}$$

$$a_{2m-1,2m-1} = 0. \tag{A2.17}$$

Again we have $a_{ij} = 0$, if $i+j$ is even, so that

$$\mathrm{Pf}[a_{ij}]_{i,j=0,1,\ldots,2m-1} = \det[a_{2i,2j+1}]_{i,j=0,1,\ldots,m-1}. \tag{A2.18}$$

We can again, without changing the value of the determinant, replace $a_{2i,2j+1}$ by b_{ij}, where for $j = 0, 1, \ldots, m-2$ and $i = 0, 1, \ldots, m-1$ the b_{ij} is given by equation (A2.11),while for $i = 0, 1, \ldots, m-1$,

$$b_{i,m-1} = \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}\alpha^2 x^2)\varphi_{2i}(x)\, \mathrm{d}x = \left(\frac{2\pi}{1+\alpha^2}\right)^{1/2} (2^{2i}(2i)!\sqrt{\pi})^{-1/2} \frac{(2i)!}{i!} \left(\frac{1-\alpha^2}{1+\alpha^2}\right)^i \tag{A2.19}$$

(equation A1.12).

Collecting the results for $N = 2m-1$, one has

$$C_N^{-1} = \prod_0^{N-1} (2^{-j} j! \sqrt{\pi})^{1/2} N! \prod_0^{m-2} \left[\frac{2}{1+\alpha^2}\left(\frac{2}{2j+1}\right)^{1/2}\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{2j+1/2}\right]$$

$$\times \left(\frac{2\pi}{1+\alpha^2}\right)^{1/2} (2^{2m-2}(2m-2)!\sqrt{\pi})^{-1/2} \frac{(2m-2)!}{(m-1)!} \left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{m-1}, \tag{A2.20}$$

which on similar manipulations can be seen to be equation (2.11) with $N = 2m-1$.

Appendix 3. Another normalisation integral

The constant in equation (2.51) being fixed by the normalisation condition

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x_1, \ldots, x_{2N})\, dx_1 \ldots dx_{2N} = 1,$$

with $p(x_1, \ldots, x_{2N})$ given by (2.49), we need the integral

$$C_N^{-1} = \int_{-\infty}^{\infty} \cdots \int dx_1 \ldots dx_{2N} \exp\left(-\frac{1+\alpha^2}{2} \sum_1^{2N} x_i^2\right) \Delta(x)\, \mathrm{Pf}[F(x_i - x_j)],$$

$$F(x) = x \exp[-(1-\alpha^4)x^2/4\alpha^2].$$

We follow the reasoning of appendix 2, case even number of variables, and write

$$C_N^{-1} \prod_0^{2N-1} (2^{-j} j! \sqrt{\pi})^{-1/2} = (2N)!\, \mathrm{Pf}[a_{ij}]_{i,j=0,1,\ldots,2N-1}$$

$$= (2N)!\, \det[a_{2i,2j+1}]_{i,j=0,1,\ldots,N-1}$$

where

$$a_{ij} = \iint_{-\infty}^{\infty} F(y-x) \exp[-\tfrac{1}{2}\alpha^2(x^2+y^2)] \varphi_i(x) \varphi_j(y)\, dx\, dy$$

$$= \iint_{-\infty}^{\infty} (y-x)\varphi_i(x)\varphi_j(y) \exp\left(-\tfrac{1}{2}\alpha^2(x^2+y^2) - \frac{1-\alpha^4}{4\alpha^2}(x-y)^2\right) dx\, dy.$$

Now from (A1.3), (A1.16) and (A1.1) one gets after a little algebra

$$a_{2i,2j+1} = \frac{2\alpha^3\sqrt{2\pi}}{(1+\alpha^2)^2} \left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{2i} (\sqrt{2i+1}\, \delta_{ij} + \sqrt{2i}\, \delta_{i,j+1}).$$

Thus

$$C_N^{-1} = \prod_0^{2N-1} (2^{-j} j! \sqrt{\pi})^{1/2} (2N)! \prod_0^{N-1} \left[\frac{2\alpha^3\sqrt{2\pi}}{(1+\alpha^2)^2} \left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{2j} \sqrt{2j+1}\right]$$

which is the same as equation (2.51).

Appendix 4. The joint probability density as a *T*det of a self-dual quaternion matrix

Equation (4.7) will be verified separtely for the cases N even and N odd.

When N is even, we observe that the $2N \times 2N$ matrix

$$G = \begin{bmatrix} S_N(x_i, x_j) & D_N(x_i, x_j) \\ I_N(x_i, x_j) & S_N(x_j, x_i) \end{bmatrix}_{i,j=1,\ldots,N,} \tag{A4.1}$$

can be written as a product of two rectangular matrices of orders $2N\times N$ and $N\times 2N$ respectively

$$G=G_1G_2, \tag{A4.2}$$

$$G_1=\begin{bmatrix}\varphi_{2k}(x_j) & -\psi_{2k}(x_i)\\ -A_{2k}(x_i) & \varphi_{2k}(x_i)\end{bmatrix}_{\substack{i=1,2,\ldots,N,\\ k=0,1,\ldots,N/2-1}},$$

$$G_2=\begin{bmatrix}\varphi_{2k}(x_j) & \psi_{2k}(x_j)\\ A_{2k}(x_j) & \varphi_{2k}(x_j)\end{bmatrix}_{\substack{k=0,1,\ldots,N/2-1\\ j=1,2,\ldots,N}}.$$

The rank of G_1 or of G_2 is at most N; it is in fact N, since the $\varphi_{2k}(x)$ is an orthonormal sequence. The rank of G, the product of G_1 and G_2, is therefore at most N (see e.g. Mehta 1977); in fact we know that it is N, since its first N rows are linearly independent. Therefore, the last N rows of G are linear combinations of its first N rows and *vice versa*. The determinant of the $2N\times 2N$ matrix

$$[\Phi(x_i,x_j)]=\begin{bmatrix}S_N(x_i,x_j) & D_N(x_i,x_j)\\ J_N(x_i,x_j) & S_N(x_j,x_i)\end{bmatrix}_{i,j=1,\ldots,N}$$

is therefore not changed if we subtract from its last N rows the corresponding rows of G:

$$\begin{aligned}\det[\Phi(x_i,x_j)]&=\det\begin{bmatrix}S_N(x_i,x_j) & D_N(x_i,x_j)\\ g(x_i,x_j) & 0\end{bmatrix}\\ &=(-1)^N\det[g(x_i,x_j)]\det[D_N(x_i,x_j)].\end{aligned} \tag{A4.3}$$

But

$$\det[g(x_i,x_j)]=2^{-N}\left(\frac{1+\alpha^2}{1-\alpha^2}\right)^{N/2}\exp\left(-\alpha^2\sum x_i^2\right)\det[f(x_i-x_j)] \tag{A4.4}$$

and

$$\begin{aligned}\det[D_N(x_i,x_j)]&=\det[\varphi_{2k}(x_i)\quad \psi_{2k}(x_i)]\begin{bmatrix}\psi_{2k}(x_j)\\ -\varphi_{2k}(x_j)\end{bmatrix}\\ &=(\det[\varphi_{2k}(x_i)\quad \psi_{2k}(x_i)])^2.\end{aligned} \tag{A4.5}$$

Now ψ_{2k} being a linear combination of φ_{2k-1} and φ_{2k+1}, equation (A1.4), we can replace the columns $\psi_0,\psi_2,\ldots,\psi_{2N-2}$ successively by $\varphi_1,\varphi_3,\ldots,\varphi_{2N-1}$ in the determinant. The result is

$$\begin{aligned}\det[D_N(x_i,x_j)]&\propto\{\det[\varphi_{j-1}(x_i)]\}^2\\ &\propto\exp\left(-\sum x_i^2\right)\{\det[x_i^{j-1}]\}^2\\ &\propto\exp\left(-\sum x_i^2\right)[\Delta(x)]^2.\end{aligned} \tag{A4.6}$$

From equations (A4.3), (A4.4) and (A4.6) we get

$$\det[\Phi(x_i,x_j)]\propto\exp\left(-(1+\alpha^2)\sum_1^N x_i^2\right)\det[f(x_i-x_j)][\Delta(x)]^2,$$

which in view of equation (2.7) is

$$\det[\Phi(x_i, x_j)] \propto [p(x_1, \ldots, x_N)]^2. \tag{A4.7}$$

When $N = 2m+1$ odd, the method used above fails at one point; the $N \times N$ matrix

$$D_N(x_i, x_j) = [\varphi_{2k+1}(x_i) \quad -\psi_{2k+1}(x_i)]\begin{bmatrix}\psi_{2k+1}(x_j)\\ \varphi_{2k+1}(x_j)\end{bmatrix} \tag{A4.8}$$

of rank $N-1$ has a zero determinant, while its multiplying factor becomes infinite due to the extra terms ξ_N and μ_N.

Instead of G consider the $2N \times 2N$ matrix

$$G_\delta = \begin{bmatrix}\varphi_{2k+1}(x_i) & -\psi_{2k+1}(x_i) & \delta\varphi_{2m}(x_i)\\ -A_{2k+1} & \varphi_{2k+1}(x_i) & (c\delta)^{-1}\exp(-\frac{1}{2}\alpha^2 x_i^2)\end{bmatrix}\begin{bmatrix}\varphi_{2k+1}(x_j) & \psi_{2k+1}(x_j)\\ A_{2k+1}(x_j) & \varphi_{2k+1}(x_j)\\ (c\delta)^{-1}\exp(-\frac{1}{2}\alpha^2 x_j^2) & \delta\varphi_{2m}(x_j)\end{bmatrix} \tag{A4.9}$$

where δ is arbitrary and

$$c = \int_{-\infty}^{\infty} \varphi_{2m}(t)\exp(-\tfrac{1}{2}\alpha^2 t^2)\,\mathrm{d}t. \tag{A4.10}$$

Thus

$$G_\delta = \begin{bmatrix} S_N(x_i, x_j) + \xi_N(x_i, x_j) & D_N(x_i, x_j) + \delta^2\varphi_{2m}(x_i)\varphi_{2m}(x_j)\\ I_N(x_i, x_j) + (c\delta)^{-2}\exp[-\frac{1}{2}\alpha^2(x_i^2 + x_j^2)] & S_N(x_j, x_i) + \xi_N(x_j, x_i)\end{bmatrix} \tag{A4.11}$$

with ξ_N, S_N, I_N and D_N given by equations (2.15), (2.17), (2.22) and (2.19). The rank of G_δ is N. The determinant of the $2N \times 2N$ matrix

$$[\Phi_\delta(x_i, x_j)] = \begin{bmatrix} S_N(x_i, x_j) + \xi_N(x_i, x_j) & D_N(x_i, x_j) + \delta^2\varphi_{2m}(x_i)\varphi_{2m}(x_j)\\ J_N(x_i, x_j) & S_N(x_j, x_i) + \xi_N(x_j, x_i)\end{bmatrix}, \tag{A4.12}$$

is not changed if we subtract from its last N rows the corresponding rows of G_δ; the resulting determinant factorises,

$$\begin{aligned}\det[\Phi_\delta(x_i, x_j)] &= \det[D_N(x_i, x_j) + \delta^2\varphi_{2m}(x_i)\varphi_{2m}(x_j)](-1)^N \det\{g(x_i, x_j)\\ &\quad - (c\delta)^{-2}\exp[-\tfrac{1}{2}\alpha^2(x_i^2 + x_j^2)] + \mu(x_i, x_j) - \mu(x_j, x_i)\}.\end{aligned} \tag{A4.13}$$

The first factor is

$$\delta^2\{\det[\varphi_1(x_i), \psi_1(x_i); \varphi_3(x_i), \psi_3(x_i; \ldots; \varphi_{2m-1}(x_i), \psi_{2m-1}(x_i); \varphi_{2m}(x_i)]\}^2.$$

We can replace the last but one column by a linear combination of the last two columns. Choosing this combination properly, equation (A1.4), ψ_{2m-1} can be replaced by φ_{2m-2}. Then the column ψ_{2m-3} can be replaced by the column φ_{2m-4}, and so on. Thus the column ψ_{2k+1} is replaced by φ_{2k} for $k = 0, 1, \ldots, m-1$, the determinant being multiplied by a constant. Thus the first factor in (A4.13) is proportional to

$$\delta^2\{\det[\varphi_{j-1}(x_i)]_{i,j=1,2,\ldots,2m+1}\}^2 \propto \delta^2 \exp\Bigl(-\sum x_i^2\Bigr)[\Delta(x)]^2. \tag{A4.14}$$

The second factor in equation (A4.13) is proportional to

$$\exp\left(-\alpha^2 \sum x_i^2\right) \det[f(x_i - x_j) - k\delta^{-2} + k'(h(x_i) - h(x_j))] \tag{A4.15}$$

where

$$h(x) = \int_{-\infty}^{\infty} \exp(\tfrac{1}{2}\alpha^2 t^2)\varepsilon(x-t)\varphi_{2m}(t)\, dt \tag{A4.16}$$

and k and k' are certain constants. But

$$\det[f(x_i - x_j) - k\delta^{-2} + k'(h(x_i) - h(x_j))]$$

$$= \det\begin{bmatrix} f(x_i - x_j) - k\delta^{-2} + k'(h(x_i) - h(x_j)) & 0 & k\delta^{-2} - k'h(x_i) \\ k\delta^{-2} + k'h(x_j) & 1 & -k\delta^{-2} \\ 0 & 0 & 1 \end{bmatrix} \tag{A4.17}$$

$$= \det\begin{bmatrix} f(x_i - x_j) & -k'h(x_i) & 1 \\ k'h(x_j) & -k\delta^{-2} & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= -k\delta^{-2} \det\begin{bmatrix} f(x_i - x_j) & 1 \\ -1 & 0 \end{bmatrix} + \det\begin{bmatrix} f(x_i - x_j) & -k'h(x_i) & 1 \\ k'h(x_j) & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}. \tag{A4.18}$$

The second determinant, being that of an anti-symmetric matrix of odd order $N+2$, is zero.

Thus the second factor of equation (A4.13) is proportional to

$$\delta^{-2} \exp(-\alpha^2 \sum x_i^2) \det\begin{bmatrix} f(x_i - x_j) & 1 \\ -1 & 0 \end{bmatrix}. \tag{A4.19}$$

From eqations (A4.13)–(A4.14) and (A4.19) on taking the limit $\delta \to 0$, we get for N odd,

$$\det[\Phi(x_i, x_j)] \propto \left[\exp\left(-\tfrac{1}{2}(1+\alpha^2) \sum x_i^2\right) \Delta(x)\right]^2 \det[F_{ij}] \propto [p(x_1, \ldots, x_N)]^2 \tag{A4.20}$$

in view of equation (2.7). The value (A4.10) of the constant c is not required for this appendix; it will be needed in appendix 5.

Thus whether N is even or odd, the determinant of the $2N \times 2N$ matrix $[\Phi(x_i, x_j)]$ is proportional to $\{p(x_1, \ldots, x_N)\}^2$, equation (A4.7) or (A4.20). Therefore, from theorem 4.2

$$T\det[\Phi(x_i, x_j)] \propto p(x_1, \ldots, x_N). \tag{A4.21}$$

The constant of proportionality is fixed by the normalisation, theorem 4.1 applied N times and the fact that

$$\int_{-\infty}^{\infty} \Phi(x, x)\, dx = \int_{-\infty}^{\infty} [S_N(x, x) + \xi_N(x, x)]\, dx = N. \tag{A4.22}$$

Arguments similar to the even-N case given above show that

$$\det[\boldsymbol{\Phi}(x_i, x_j)] \propto [\boldsymbol{p}(x_1, \ldots, x_{2N})]^2, \tag{A4.23}$$

which is equation (4.9) except for the constant. This constant is again determined by normalisation, theroem 4.1, and the fact that

$$\int_{-\infty}^{\infty} \boldsymbol{\Phi}(x, x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \boldsymbol{S}_N(x, x)\,\mathrm{d}x = 2N. \tag{A4.24}$$

Appendix 5. Verification of equation (4.8)

We will verify the equation

$$\int_{-\infty}^{\infty} \Phi(x, y)\Phi(y, z)\,\mathrm{d}y = \Phi(x, z) + \tau\Phi(x, z) - \Phi(x, z)\tau, \tag{4.8}$$

where Φ is defined by equation (2.14) and

$$\tau = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}. \tag{A5.1}$$

We will also verify the same equation when Φ is replaced by $\boldsymbol{\Phi}$, equation (2.52), and τ by $-\tau$.

Writing for brevity $F^{\dagger}(x, y) = F(y, x)$ and

$$F * G \equiv \int_{-\infty}^{\infty} F(x, y)G(y, z)\,\mathrm{d}y, \tag{A5.2}$$

equation(4.8) above is

$$\Phi * \Phi \equiv \begin{bmatrix} S_N + \xi_N & D_N \\ J_N & S_N^{\dagger} + \xi_N^{\dagger} \end{bmatrix} * \begin{bmatrix} S_N + \xi_N & D_N \\ J_N & S_N^{\dagger} + \xi_N^{\dagger} \end{bmatrix} = \begin{bmatrix} S_N + \xi_N & 2D_N \\ 0 & S_N^{\dagger} + \xi_N^{\dagger} \end{bmatrix}, \tag{A5.3}$$

where S_N, ξ_N, D_N and J_N are defined by (2.17), (2.15), (2.19) and (2.20). This amounts to verifying that

$$(S_N + \xi_N) * (S_N + \xi_N) = S_N + \xi_N, \qquad D_N * J_N = 0 = J_N * (S_N + \xi_N),$$
$$(S_N + \xi_N) * D_N = D_N, \tag{A5.4}$$

and, as a consequence,

$$(S_N + \xi_N)^{\dagger} * (S_N + \xi_N)^{\dagger} = (S_N + \xi_N)^{\dagger}, \qquad J_N * D_N = 0 = (S_N + \xi_N)^{\dagger} * J_N,$$
$$D_N * (S_N + \xi_N)^{\dagger} = D_N. \tag{A5.5}$$

For the verification of (A5.4) we will repeatedly use results of orthonormality and convolution integrals of appendix 1, §§ A1.3 and A1.4. Thus from the expressions (2.17), (2.19) and (2.22) and the equations of appendix A1.3, we have

$$S_N * S_N = S_N, \qquad S_N * D_N = D_N,$$
$$D_N * I_N = S_N, \qquad I_N * S_N = I_N, \tag{A5.6}$$

while from appendix A1.4 we have

$$g * S_N = -I_N, \qquad D_N * g = -S_N. \tag{A5.7}$$

For N odd we also need

$$\xi_N * \xi_N = \xi_N, \qquad \mu_N^{\dagger} * \xi_N = \mu_N^{\dagger}, \tag{A5.8}$$

$$g * \xi_N = \mu_N^{\dagger}, \tag{A5.9}$$

and

$$S_N * \xi_N, \qquad \xi_N * S_N, \qquad \xi_N * D_N, \qquad \mu_N * \xi_N,$$
$$(\mu_N - \mu_N^{\dagger}) * S_N, \qquad D_N * (\mu_N - \mu_N^{\dagger}). \tag{A5.10}$$

Those in (A5.8) are easily verified, and (A5.9) is a consequence of (A1.15). For (A5.10) we need to know

$$\int_{-\infty}^{\infty} \varphi_{2j+1}(y)\varphi_{2m}(y)\,dy, \qquad \int_{-\infty}^{\infty} \psi_{2j+1}(y)A_{2m}(y)\,dy,$$
$$\int_{-\infty}^{\infty} \varphi_{2j+1}(y)\exp(-\tfrac{1}{2}\alpha^2 y^2)\,dy, \qquad \int_{-\infty}^{\infty} A_{2m}(y)\varphi_{2m}(y)\,dy, \tag{A5.11}$$

and

$$\int_{-\infty}^{\infty} A_{2j+1}(y)\varphi_{2m}(y)\,dy, \qquad \int_{-\infty}^{\infty} \varphi_{2j+1}(y)A_{2m}(y)\,dy, \tag{A5.12}$$

$$\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}\alpha^2 y^2)\psi_{2j+1}(y)\,dy. \tag{A5.13}$$

Integrals (A5.11) are zero by parity. For (A5.12) we note that A_{2j+1} is a linear combination of φ_{2j} and A_{2j-1}, equation (A1.5), and hence of $\varphi_{2j}, \varphi_{2j-2}, \ldots, \varphi_2, \varphi_0$. Similarly φ_{2j+1} is a linear combination of $\psi_{2j}, \psi_{2j-2}, \ldots, \psi_2, \psi_0$ by equation (A1.4). Thus A_{2j+1} is orthogonal to φ_{2m} and φ_{2j+1} is orthogonal to A_{2m}, for $j<m$ (see appendix A1.3). The integrand in (A5.13) is a perfect derivative, equation (2.27), of a quantity vanishing at both ends.

Therefore all the integrals in (A5.11)–(A5.13) are zero and so are (A5.10).

Thus we have verified equation (4.8) in all the cases for $\Phi(x, y)$ given by equation (2.14).

The verification of (4.8) with Φ replaced by $\boldsymbol{\Phi}$, equation (2.52), and τ replaced by $-\tau$ is similar. From the expressions (2.53), (2.55), (2.58) and (2.60) together with (A1.1), (A1.21)–(A1.24) we have

$$\boldsymbol{S}_N * \boldsymbol{S}_N = \boldsymbol{S}_N \qquad \boldsymbol{S}_N * \boldsymbol{D}_N = \boldsymbol{D}_N, \qquad \boldsymbol{I}_N * \boldsymbol{S}_N = \boldsymbol{I}_N,$$
$$\boldsymbol{S}_N * \boldsymbol{g} = -\boldsymbol{D}_N, \qquad \boldsymbol{D}_N * \boldsymbol{I}_N = \boldsymbol{S}_N, \qquad \boldsymbol{g} * \boldsymbol{I}_N = \boldsymbol{S}_N,$$

or

$$\boldsymbol{\Phi} * \boldsymbol{\Phi} \equiv \begin{bmatrix} \boldsymbol{S}_N & \boldsymbol{K}_N \\ \boldsymbol{I}_N & \boldsymbol{S}_N^{\dagger} \end{bmatrix} * \begin{bmatrix} \boldsymbol{S}_N & \boldsymbol{K}_N \\ \boldsymbol{I}_N & \boldsymbol{S}_N^{\dagger} \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_N & 0 \\ 2\boldsymbol{I}_N & \boldsymbol{S}_N^{\dagger} \end{bmatrix} = \boldsymbol{\Phi} + \boldsymbol{\Phi}\tau - \tau\boldsymbol{\Phi},$$

with τ given by (A5.1).

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† In this paper equation (3.10) should contain an extra term $\int_x^y \varphi_{2m}(t)\,dt/\int_{-\infty}^{\infty} \varphi_{2m}(t)\,dt$ for $N = 2m + 1$ odd.